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Gravity and the Standard Model with 130 GeV Truth Quark from $D_4 - D_5 - E_6$ Model using 3×3 Octonion Matrices.

Frank D. (Tony) Smith, Jr.
e-mail: gt0109e@prism.gatech.edu
and fsmith@pinet.aip.org
P. O. Box for snail-mail:
P. O. Box 430, Cartersville, Georgia 30120 USA
WWW URL http://www.gatech.edu/tsmith/home.html

School of Physics
Georgia Institute of Technology
Atlanta, Georgia 30332

Abstract

The $D_4 - D_5 - E_6$ model of gravity and the Standard Model with a 130 GeV truth quark is constructed using 3×3 matrices of octonions. The model has both continuum and lattice versions. The lattice version uses HyperDiamond lattice structure.

Contents

Foreword.

In the $D_4 - D_5 - E_6$ mode, all tree level force strengths, particle masses, and K-M parameters can be calculated. They can be found at
WWW URL http://www.gatech.edu/tsmith/home.html [71].

Within 10 % or so, they are consistent with currently accepted experimental values except for the truth quark mass. Acccording to the $D_4 - D_5 - E_6$ model, it should be 130 GeV at tree level, whereas CDF at Fermilab has interpreted their data to show that the truth quark mass is about $174 GeV$.

Fermilab's announcement is at
WWW URL http://fnnews.fnal.gov/ [26].

My opinion is that the CDF interpretation is wrong, and that currently available experimental data indicates a truth quark mass in the range of $120 - 145$ GeV, which is consistent with the 130 GeV tree level calculation of the $D_4 - D_5 - E_6$ model.

Details of my opinion about the truth quark mass are at
WWW URL http://www.gatech.edu/tsmith/TCZ.html [71].

The $D_4 - D_5 - E_6$ model uses 3 copies of the octonions to represent the 8 first generation fermion particles, the 8 first generation fermion antiparticles, and the 8-dimensional (before dimensional reduction) spacetime.

This paper is an effort to describe the construction of the $D_4 - D_5 - E_6$ model by starting with elementary structures, and building the model up from them so that the interrelationships of the parts of the model may be more clearly understood.

The starting points I have chosen are the octonions, and 3×3 matrices of octonions [25].

Since the properties of octonions are not well covered in most textbooks, the first section of the paper is devoted to a brief summary of some of the properties used in this paper.

3×3 matrices are chosen because matrices are well covered in conventional math and physics texts, and also
because 3×3 traceless octonion matrices can be used to build up in a concrete way the Lie algebras, Jordan algebras, and symmetric spaces that are needed to construct the $D_4 - D_5 - E_6$ model.

In particular, 3×3 traceless octonion matrices can be divided into hermitian and antihermitian parts to form a Jordan algebra $J_3^{\mathbf{O}}o$ and (when combined with derivations) a Lie algebra F_4.

Then $J_3^{\mathbf{O}}o$ and F_4 can be combined to form the Lie algebra E_6.

The components of the matrix representation of E_6 can then be used to construct the Lagrangian of the $D_4 - D_5 - E_6$ model of Gravity plus the Standard Model for first generation fermions.

By extension to the exceptional Lie algebras E_7 and E_8, the model includes the second and third generation fermions.

Since the E-series of Lie algebras contains only E_6, E_7 and E_8, there are only three generation of fermions in the model.

To get an overview of what this paper does, represent the E_6 Lie algebra by

$$\mathbf{R} \otimes \left(\begin{pmatrix} S^7_1 & \mathbf{O}_+ & \mathbf{O}_v \\ -\mathbf{O}^\dagger_+ & S^7_2 & \mathbf{O}_- \\ -\mathbf{O}^\dagger_v & -\mathbf{O}^\dagger_- & -S^7_1 - S^7_2 \end{pmatrix} \oplus G_2 \oplus \begin{pmatrix} a & \mathbf{O}_+ & \mathbf{O}_v \\ \mathbf{O}^\dagger_+ & b & \mathbf{O}_- \\ \mathbf{O}^\dagger_v & \mathbf{O}^\dagger_- & -a-b \end{pmatrix} \right) \quad (1)$$

where the $\mathbf{R}$ represents the real scalar field of this representation of E_6; $\mathbf{O}$ is octonion; S^7 represents the imaginary octonions; a, b, c are real numbers (octonion real axis); and G_2 is the Lie algebra of derivations of $\mathbf{O}$.

Then, use the components of E_6 to construct the 8-dimensional Lagrangian of the $D_4 - D_5 - E_6$ model:

$$\int_{V_8} F_8 \wedge \star F_8 + \partial^2_8 \overline{\Phi_8} \wedge \star \partial^2_8 \Phi_8 + \overline{S_{8\pm}} \not\partial_8 S_{8\pm} \quad (2)$$

where $\star$ is the Hodge dual;

∂_8 is the 8-dimensional covariant derivative,
$\not\partial_8$ is the 8-dimensional Dirac operator, and

F_8 is the 28-dimensional $Spin(8)$ curvature,
which come from the $Spin(0, 8)$ gauge group subgroup of E_6, represented here by

$$\begin{pmatrix} S^7_1 & 0 \\ 0 & S^7_2 \end{pmatrix} \oplus G_2 \quad (3)$$

Φ_8 is the 8-dimensional scalar field, which comes from the $\mathbf{R}$ scalar part of the representation of E_6;

V_8 is 8-dimensional spacetime, and which comes from the $\mathbf{O}_v$ part of the representation of E_6;

$S_{8\pm}$ are the $+$ and $-$ 8-dimensional half-spinor fermion spaces, which come from the $\mathbf{O}_+$ and $\mathbf{O}_-$ parts of the representation of E_6;

As a theory with an 8-dimensional spacetime, the $D_4 - D_5 - E_6$ model is seen to be constructed from the fundamental representations of the D_4 Lie algebra $Spin(0, 8)$:

Φ_8 comes from the trivial scalar representation;

F_8 comes from the 28-dimensional adjoint representation;

V_8 comes from the 8-dimensional vector representation; and

S_{8+} and S_{8-} come from the two 8-dimensional half-spinor representations.

As all representations of $Spin(0, 8)$ can be built by tensor products and sums using the 3 8-dimensional representations and the 28-dimensional adjoint representation,
(the 4 of which make up the D_4 Dynkin diagram,
which looks like a Mercedes-Benz 3-pointed star,
with the 28-dimensional adjoint representation in the middle)
plus the trivial scalar representation,
the $D_4 - D_5 - E_6$ model effectively uses all the ways you can look at $Spin(0, 8)$.

If you use exterior wedge products as well as tensor products and sums, you can build all the representations from only the representations on the exterior of the Dynkin diagram,
in this case, the 3 points of the star, the 3 8-dimensional vector and half-spinor representations,
plus the trivial scalar representation.
In this case, it is clear because the adjoint representation is the bivector rep-

resentation, or the wedge product of two copies of the vector representation. For discussion of more complicated Lie algebras, such as the E_8 Lie algebra, see Adams [2].

Our physical spacetime is not 8-dimensional, and the $D_4 - D_5 - E_6$ model gets to a 4-dimensional spacetime by a process of dimensional reduction.

Dimensional reduction of spacetime to 4 dimensions produces a realistic 4-dimensional Lagrangian of Gravity plus the Standard Model.

Then force strength constants and particle masses are calculated, as are Kobayashi-Maskawa parameters.

(The calculations are at tree level, with quark masses being constituent masses.)

The structures in this paper are exceptional in many senses, and can be studied from many points of view.

This paper is based on the structure of 3×3 matrices of octonions.
For a discussion of 3×3 matrices of octonions from a somewhat different perspective, see Truini and Biedenharn [75].

I have also looked at the $D_4 - D_5 - E_6$ model from the Clifford algebra point of view [68].
Some useful references to Clifford algebras include the books of Gilbert and Murray [31], of Harvey [37], and of Porteous [54].

From the point of view of the exceptional Lie algebra F_4 and the Cayley Moufang plane $\mathbf{O}P^2$, see the paper of Adams [1].

From the point of view of the 7-sphere, the highest dimensional sphere that is parallelizable, and the only parallelizable manifold that is not a Lie group, see papers by Cederwall and Preitschopf [6], Cederwall [7], Manogue and Schray [45], and Schray and Manogue [63].

From the point of view of Hermitian Jordan Triple systems, see papers by Günaydin [35, 36] and my paper [67].

General references for differential geometry, Lie groups, and the symmetric spaces used herein are the books of Besse [4], Fulton and Harris [28], Gilmore [32], Helgason [38], Hua [40], Kobayashi and Nomizu [42], Porteous [54], and Postnikov [55], and the papers of Ramond [56] and Sudbery [73]. An interesting discussion of symmetries in physics is Saller [57].

A notational comment - this paper uses the same notation for a Lie group as for its Lie algebra. It should be clear from context as to which is being discussed.

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1 Octonions.

There are 480 different ways to write a multiplication table for an octonion product. Since the purpose of this paper is to construct a concrete representation of the $D_4 - D_5 - E_6$ physics model, one of these is chosen and used throughout.

For a good introduction to octonions, see the books of Geoffrey Dixon [18] and of Jaak Lohmus, Eugene Paal, and Leo Sorgsepp [43], as well as the paper of Günaydin and Gürsey [34].

For many more interesting things about octonions, see the book and papers of Geoffrey Dixon [15, 16, 17, 18, 19, 20, 21].

The following description of octonion products, left and right actions, and automorphisms and derivations, is taken from Dixon's book and papers cited above, and from a paper by A. Sudbery [73].

1.1 Octonion Product.

For concreteness, choose one of the 480 multiplications: Let $e_a, a = 1, ..., 7$, represent the imaginary units of $\mathbf{O}$, and adopt the cyclic multiplication rule

$$e_a e_{a+1} = e_{a+5} = e_{a-2}, \tag{4}$$

a=1,...,7, all indices modulo 7, from 1 to 7 (another cyclic multiplication rule for $\mathbf{O}$, dual to that above, is $e_a e_{a+1} = e_{a+3} = e_{a-4}$). In particular,

$$\{q_1 \rightarrow e_a, q_2 \rightarrow e_{a+1}, q_3 \rightarrow e_{a+5}\} \tag{5}$$

define injections of $\mathbf{Q}$ into $\mathbf{O}$ for a=1,...,7. In the multiplication rule Equation (4) the indices range from 1 to 7, and the index 0 representing the octonion real number 1 is not subject to the rule.
This octonion multiplication has some very nice properties.

For example,

$$\text{if } e_a e_b = e_c, \text{ then } e_{(2a)} e_{(2b)} = e_{(2c)}. \tag{6}$$

Equation (6) in combination with Equation (4) immediately implies

$$\begin{aligned} e_a e_{a+2} &= e_{a+3}, \\ e_a e_{a+4} &= e_{a+6} \end{aligned} \tag{7}$$

(so $e_a e_{a+2^n} = e_{a-2^{n+1}}$, or $e_a e_{a+b} = [b^3 \bmod 7] e_{a-2b^4}$, $b = 1, ..., 6$, where b^3 out front provides the sign of the product (modulo 7, $1^3 = 2^3 = 4^3 = 1$, and $3^3 = 5^3 = 6^3 = -1$)). Also, 2(7)=7 mod 7, so Equations (6) and (4) imply

$$e_7 e_1 = e_5, \quad e_7 e_2 = e_3, \quad e_7 e_4 = e_6. \tag{8}$$

These modulo 7 periodicity properties are reflected in the full multiplication table:

$$\begin{pmatrix} 1 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\ e_1 & -1 & e_6 & e_4 & -e_3 & e_7 & -e_2 & -e_5 \\ e_2 & -e_6 & -1 & e_7 & e_5 & -e_4 & e_1 & -e_3 \\ e_3 & -e_4 & -e_7 & -1 & e_1 & e_6 & -e_5 & e_2 \\ e_4 & e_3 & -e_5 & -e_1 & -1 & e_2 & e_7 & -e_6 \\ e_5 & -e_7 & e_4 & -e_6 & -e_2 & -1 & e_3 & e_1 \\ e_6 & e_2 & -e_1 & e_5 & -e_7 & -e_3 & -1 & e_4 \\ e_7 & e_5 & e_3 & -e_2 & e_6 & -e_1 & -e_4 & -1 \end{pmatrix}. \tag{9}$$

Although the octonion product is nonassociative, it is alternative.

An example of nonassociativity from the multiplication table is that

$$(e_1 e_2) e_4 = e_6 e_2 = -e_7$$

is not equal to

$$e_1 (e_2 e_4) = e_1 e_5 = e_7$$

The alternativity property is the fact that the associator

$$[x, y, z] = x(yz) - (xy)z \tag{10}$$

is an alternating function of $x, y, z \in \mathbf{O}$.

1.2 Left and Right Adjoint Algebras of O.

The octonion algebra is nonassociative and so is not representable as a matrix algebra.

However, the adjoint algebras of left and right actions of **O** on itself are associative.

For example, let $u_1, ..., u_n, x$ be elements of **O**. Consider the left adjoint map

$$x \to u_n(...(u_2(u_1 x))...). \tag{11}$$

The nesting of parentheses forces the products to occur in a certain order, hence this algebra of left-actions is trivially associative, and it is representable by a matrix algebra.

One such representation can be derived immediately from the multiplication table Equation (9). For example, the actions

$$x \to e_1 x \equiv e_{L1}(x), \text{ and } x \to x e_1 \equiv e_{R1}(x) \tag{12}$$

can be identified with the matrices

$$e_{L1} \to \begin{pmatrix} . & -1 & . & . & . & . & . & . \\ 1 & . & . & . & . & . & . & . \\ . & . & . & . & . & . & -1 & . \\ . & . & . & . & -1 & . & . & . \\ . & . & . & 1 & . & . & . & . \\ . & . & . & . & . & . & . & -1 \\ . & . & 1 & . & . & . & . & . \\ . & . & . & . & . & 1 & . & . \end{pmatrix}, \tag{13}$$

$$e_{R1} \to \begin{pmatrix} . & -1 & . & . & . & . & . & . \\ 1 & . & . & . & . & . & . & . \\ . & . & . & . & . & . & 1 & . \\ . & . & . & . & 1 & . & . & . \\ . & . & . & -1 & . & . & . & . \\ . & . & . & . & . & . & . & 1 \\ . & . & -1 & . & . & . & . & . \\ . & . & . & . & . & -1 & . & . \end{pmatrix} \tag{14}$$

(only nonzero entries are indicated). Note that $e_2e_6 = e_3e_4 = e_5e_7 = e_1$, but because of the nonassociativity of **O**, for example,

$$e_1x = (e_2e_6)x \neq e_2(e_6x) \equiv e_{L26}(x) \tag{15}$$

in general.

In the octonion algebra, any product from the right can be reproduced as the sum of products from the left, and visa versa.

The left and right adjoint algebras $\mathbf{O}_L$ and $\mathbf{O}_R$ are the same algebra, and this algebra is larger than **O** itself. In fact, it is isomorphic to $\mathbf{R}(8)$.

It is not difficult to prove that

$$e_{La...bc...d} = -e_{La...cb...d} \tag{16}$$

if $b \neq c$, all indices from 1 to 7.

So,for example, $e_1(e_2(e_3x)) = -e_1(e_3(e_2x)) = e_3(e_1(e_2x))$.

In addition,

$$e_{Lab...pp...c} = -e_{Lab...c} \tag{17}$$

(cancellation of like indices).

Together with

$$e_{L7654321} = \mathbf{1}, \tag{18}$$

Equations (16) and (17) imply that a complete basis for the left=right adjoint algebra of **O** consists of elements of the form

$$\mathbf{1}, e_{La}, e_{Lab}, e_{Labc}, \tag{19}$$

1 the identity.

This yields 1+7+21+35=64 as the dimension of the adjoint algebra of **O**, also the dimension of $\mathbf{R}(8)$.

$$\mathbf{O}_L = \mathbf{O}_R = \mathbf{R}(8). \tag{20}$$

The 8-dimensional **O** itself is the object space of the adjoint algebra.

1.3 Aut(O) and Der(O).

The 14-dimensional exceptional Lie group G_2 is the automorphism group $Aut(\mathbf{O})$ of the octonions.

When viewed from the point of view of a linear Lie algebra with bracket product rather than the point of view of a nonlinear global Lie group with group product, the structure that corresponds to the automorphism group $Aut(\mathbf{O})$ is the algebra of derivations $Der(\mathbf{O})$ of the octonions.

A derivation is a linear map $D : \mathbf{O} \to \mathbf{O}$ such that for $x, y \in \mathbf{O}$:

$$D(xy) = (Dx)y + x(Dy) \tag{21}$$

Then, from the alternative property of the octonions:

$$D(ab)x = [a, b, x] + (1/3)[[a, b], x] \tag{22}$$

Let $C_d = L_d - R_d$, where L_d and R_d denote left and right multiplication by an imaginary octonion d. The imaginary octonions $Im(\mathbf{O})$ are those octonions in the space orthogonal to the octonion real axis. Then

$$D(ab) = (1/6)([C_a, C_b] + C_{[a,b]})$$

The algebra of derivations $Der(\mathbf{O})$ of the octonions is the Lie algebra G_2.

A basis for the Lie algebra of G_2 is represented in $\mathbf{O}_L$ by:

$$\{e_{Lab} - e_{Lcd} : e_a e_b = e_c e_d\} \to G_2. \tag{23}$$

In $\mathbf{O}_R$ the basis is much the same:

$$\{e_{Rab} - e_{Rcd} : e_a e_b = e_c e_d\} \to G_2.$$

The stability group of any fixed octonion direction is the 8-dimensional $SU(3) \subset G_2$.

A basis for the Lie algebra of the stability group of e_7 is:

$$\{e_{Lab} - e_{Lcd} \in G_2 : a, b, c, d \neq 7\} \to su(3). \tag{24}$$

Thus $SU(3)$ is the intersection of G_2 with $Spin(6)$, and
$G_2 = SU(3) \oplus S^6$.

The algebra of derivations does not give the Lie algebra all the antisymmetric maps of a real division algebra $\mathbf{R}, \mathbf{C}, \mathbf{H}, \mathbf{O}$ unless the algebra is commutative and associative, i.e., $\mathbf{R}, \mathbf{C}$. In the case of $\mathbf{C}$, the Lie algebra of all antisymmetric maps is

$$Der(\mathbf{C}) = Spin(2) = U(1)$$

.

For $\mathbf{H}$, which is associative but not commutative,

$$Der(\mathbf{H}) \subset L_{Im(\mathbf{H})} \oplus R_{Im(\mathbf{H})}$$

where $L_{Im(\mathbf{H})}$ and $R_{Im(\mathbf{H})}$ denote left and right multiplication on the quaternions by the imaginary quaternions.
$L_{Im(\mathbf{H})}$ is isomorphic to and commutes with $R_{Im(\mathbf{H})}$.
Since $L_{Im(\mathbf{H})} = R_{Im(\mathbf{H})} = Spin(3) = SU(2) = Sp(1) = S^3$, the Lie algebra $Spin(4)$ of all antisymmetric maps of $\mathbf{H}$ is given by

$$Spin(4) = Spin(3) \oplus Spin(3)$$

For $\mathbf{O}$, which is neither associative nor commutative, the vector space of all antisymmetric maps of $\mathbf{O}$ is given by

$$Spin(0,8) = Der(\mathbf{O}) \oplus L_{Im(\mathbf{O})} \oplus R_{Im(\mathbf{O})} = G_2 \oplus S^7 \oplus S^7$$

where S^7 represents the imaginary octonions, notation suggested by the fact that the unit octonions are the 7-sphere S^7 which, since it is parallelizable, is locally representative of the imaginary octonions.

The 7-sphere S^7 has a left-handed basis $\{e_{La}\}$ and
a right-handed basis $\{e_{Ra}\}$.

Unlike the 3-sphere, which is the Lie group $Spin(3) = SU(2) = Sp(1)$, S^7 does not close under the commutator bracket product because $[e_{La}, e_{Lb}]/2 = e_{Lab}$ and $[e_{Ra}, e_{Rb}]/2 = e_{Rab}$ for $a \neq b$.

To make a Lie algebra out of S^7, it must be extended to $Spin(0, 8)$ by adding the 21 basis elements $\{e_{Lab}\}$ or $\{e_{Rab}\}$.

There are three ways to extend S^7 to the Lie algebra $Spin(0, 8)$.

They result in the left half-spinor, right half-spinor, and vector representations of $Spin(0, 8)$:

$$\begin{gathered} \{e_{La}, e_{Lbc}\} \to Spin(0,8), \text{left half-spinor}, \\ \{e_{Ra}, -e_{Rbc}\} \to Spin(0,8), \text{right half-spinor}, \\ \{e_{La} + e_{Ra}, e_{Lbc} - e_{Ra} : e_a = e_b e_c\} \to Spin(0,8), \text{vector}. \end{gathered} \tag{25}$$

Therefore, the two half-spinor representations and the vector representation of $Spin(0, 8)$ all have 8-dimensional $\mathbf{O}$ as representation space.

The three representations are isomorphic by triality.

Consider the 28 basis elements $\{e_{La} + e_{Ra}, e_{Lbc} - e_{Ra} : e_a = e_b e_c\}$ of the vector representation of $Spin(0, 8)$:
The 21-element subset $\{e_{Lbc} - e_{Ra} : e_a = e_b e_c\}$ is a basis
for the Lie algebra $Spin(7) = G_2 \oplus S^7$.

Therefore, the Lie algebra $Spin(0, 8) = S^7 \oplus G_2 \oplus S^7$.

2 3×3 Octonion Matrices and E_6.

The $D_4 - D_5 - E_6$ model of physics uses 3 copies of the octonions:
$\mathbf{O_v}$ to represent an 8-dimensional spacetime (prior to dimensional reduction to 4 dimensions);
$\mathbf{O}_+$ to represent the 8 first-generation fermion +half-spinor particles ; and
$\mathbf{O}_-$ to represent the 8 first-generation fermion -half-spinor antiparticles.

Consider the 72-dimensional space of 3×3 matrices of octonions:

$$\begin{pmatrix} \mathbf{O}_1 & \mathbf{O}_+ & \mathbf{O}_v \\ \mathbf{X} & \mathbf{O}_2 & \mathbf{O}_- \\ \mathbf{Z} & \mathbf{Y} & \mathbf{O}_3 \end{pmatrix} \tag{26}$$

where
$\mathbf{O}_v, \mathbf{O}_+, \mathbf{O}_-, \mathbf{O}_1, \mathbf{O}_2, \mathbf{O}_3, \mathbf{X}, \mathbf{Y}, \mathbf{Z}$ are octonion,
$\mathbf{O}_1 = a + S^7_1, \mathbf{O}_2 = b + S^7_2, \mathbf{O}_3 = c + S^7_3$
a, b, c are real, and
S^7_1, S^7_2, S^7_3 are imaginary octonion.

Consider the ordinary matrix product AB of two 3×3 octonion matrices A and B.

Now, to construct the Lie algebra E_6 from 3×3 octonion matrices, it is useful to split the product AB into antisymmetric and symmetric parts.

$$AB = (1/2)(AB - BA) + (1/2)(AB + BA) \tag{27}$$

This will enable us to construct an F_4 Lie algebra from antiHermitian matrices that will arise from considering the antisymmetric product, and a $J_3^{\mathbf{O}} o$ Jordan algebra from hermitian matrices that will arise from considering the symmetric product.

Then the Lie algebra F_4 and the Jordan algebra $J_3^{\mathbf{O}}o$ will be combined to form the Lie algebra E_6 whose structure forms the basis of the $D_4 - D_5 - E_6$ model.

This construction is only possible in this case because of many exceptional structures and symmetries. The $D_4 - D_5 - E_6$ model therefore inherits remarkable symmetry structures

2.1 Antihermitian Matrices and Lie Algebras.

Consider the antisymmetric product $(1/2)(AB - BA)$:
The 45-dimensional space of antihermitian 3×3 octonion matrices does not close under the antisymmetric product to form a Lie algebra (here, † denotes octonion conjugation):

$$\begin{pmatrix} S_1^7 & \mathbf{O}_+ & \mathbf{O}_v \\ -\mathbf{O}_+^\dagger & S_2^7 & \mathbf{O}_- \\ -\mathbf{O}_v^\dagger & -\mathbf{O}_-^\dagger & S_3^7 \end{pmatrix} \tag{28}$$

A product that closes is

$$(1/2)(AB - BA - Tr(AB - BA)) \tag{29}$$

The form of the product indicates that to get closure, you have to use only the 45-7 = 38-dimensional space of traceless antihermitian 3×3 octonion matrices:

$$\begin{pmatrix} S_1^7 & \mathbf{O}_+ & \mathbf{O}_v \\ -\mathbf{O}_+^\dagger & S_2^7 & \mathbf{O}_- \\ -\mathbf{O}_v^\dagger & -\mathbf{O}_-^\dagger & -S_1^7 - S_2^7 \end{pmatrix} \tag{30}$$

However, the Jacobi identity is not satisfied, so you still do not have a Lie algebra.

As was mentioned in Section 1.3, to get the Lie algebra of all antisymmetric maps of the nonassociative, noncommutative real division algebra **O** you must include the 14-dimensional Lie algebra of derivations $Der(\mathbf{O}) = G_2$.

Adding the derivations to the product that closes gives a product that not only closes but also satisfies the Jacobi identity:

$$(1/2)(AB - BA - Tr(AB - BA) \oplus D(A, B)) \tag{31}$$

where the derivation $D(A,B) = \sum_{ij} D(a_{ij}, b_{ij})$
and the derivation $D(x,y)$ acts on the octonion z
by using the alternator $[x,y,z] = D(x,y)z$.

The resulting space is the 38+14 = 52-dimensional Lie algebra F_4.

$$\begin{pmatrix} S_1^7 & \mathbf{O}_+ & \mathbf{O}_v \\ -\mathbf{O}_+^\dagger & S_2^7 & \mathbf{O}_- \\ -\mathbf{O}_v^\dagger & -\mathbf{O}_-^\dagger & -S_1^7 - S_2^7 \end{pmatrix} \oplus G_2 \tag{32}$$

The physical interpretation of this representation of F_4 in the $D_4 - D_5 - E_6$ model is:

$\mathbf{O}_v$ is 8-dimensional spacetime before dimensional reduction to 4 dimensions;

$\mathbf{O}_+$ is the 8-dimensional space representing the first generation fermion particles (the neutrino, the electron, the red, blue and green up quarks, and the red, blue and green down quarks);

$\mathbf{O}_-$ is the 8-dimensional space representing the first generation fermion antiparticles before dimensional reduction creates 3 generations; and

$S_1^7 \oplus S_2^7 \oplus G_2$ is the $Spin(0,8)$ gauge group before dimensional reduction breaks it down into gravity plus the Standard Model.

Now, consider the symmetric Jordan product and the hermitian matrices that form an algebra under it.

In the next subsection, they will be studied so that their structure can be added to the F_4 structure of traceless antihermitian matrices plus the derivations G_2.

2.2 Hermitian Matrices and Jordan Algebras.

Consider the symmetric product $(1/2)(AB + BA)$:

The 27-dimensional space of hermitian 3×3 octonion matrices closes under the symmetric product to form the Jordan algebra J_3^O:

$$\begin{pmatrix} a & \mathbf{O}_+ & \mathbf{O}_v \\ \mathbf{O}_+^\dagger & b & \mathbf{O}_- \\ \mathbf{O}_v^\dagger & \mathbf{O}_-^\dagger & c \end{pmatrix} \tag{33}$$

Even though the full 27-dimensional space of hermitian matrices forms a Jordan algebra J_3^O, only the 26-dimensional traceless subalgebra $J_3^O o$ is acted on by F_4 as its representation space:

$$\begin{pmatrix} a & \mathbf{O}_+ & \mathbf{O}_v \\ \mathbf{O}_+^\dagger & b & \mathbf{O}_- \\ \mathbf{O}_v^\dagger & \mathbf{O}_-^\dagger & -a-b \end{pmatrix} \tag{34}$$

The 52-dimensional F_4 and the 26-dimensional $J_3^O o$ combine to form the 78-dimensional Lie algebra E_6,

$$\begin{pmatrix} S_1^7 & \mathbf{O}_+ & \mathbf{O}_v \\ -\mathbf{O}_+^\dagger & S_2^7 & \mathbf{O}_- \\ -\mathbf{O}_v^\dagger & -\mathbf{O}_-^\dagger & -S_1^7 - S_2^7 \end{pmatrix} \oplus G_2 \oplus \begin{pmatrix} a & \mathbf{O}_+ & \mathbf{O}_v \\ \mathbf{O}_+^\dagger & b & \mathbf{O}_- \\ \mathbf{O}_v^\dagger & \mathbf{O}_-^\dagger & -a-b \end{pmatrix} \tag{35}$$

the E_6 of the $D_4 - D_5 - E_6$ model.

E_6 preserves the cubic determinant pseudoscalar 3-form for 3×3 octonionic matrices (see [28, 8]).

Sudbery [73] has identified E_6 with $SL(3, \mathbf{O})$.

Flynn [25], in the context of her physics models,
has used such an identification to note
the similarity of E_6 to $SL(3, \mathbf{C})$,
which is also made up of an antisymmetric part, $SU(3)$,
plus a symmetric part, an 8-dimensional Jordan algebra,
and which preserves a cubic determinant.

3 Shilov Boundaries of Complex Domains.

In the $D_4 - D_5 - E_6$ model, physical spacetime and the physical spinor fermion representation manifold are Shilov boundaries of bounded complex domains.

The best general reference to Shilov boundaries is Helgason's 1994 book [39].

The best set of calculations of volumes, etc., of Shilov boundaries is Hua's book [40].

That means, for instance, that physical 8-dimensional spacetime is the 8-real dimensional Shilov boundary of of a 16-real dimensional (8-complex-dimensional) bounded complex domain.

To physicists, the most familiar example of bounded complex domains and their Shilov boundaries (other than the unit disk and its Shilov boundary, the circle) probably comes from the twistors of Roger Penrose [53].
Another example, possibly less familiar, is the chronometry theory of I. E. Segal [64] at M.I.T.
Still another example, also probably not very familiar, is the use of the geometry of bounded complex domains by Armand Wyler [77] in his effort to calculate the value of the fine structure constant.

To mathematicians, such structures are well known.
A standard general reference is the book of Hua [40], in which Shilov boundaries are called characteristic manifolds.
Actually, I would prefer the term characteristic boundary, because it would describe it as being part of a boundary that characterizes important structures on the manifold.
However, I will use the term Shilov boundary because that seems to be the dominant term in English-language literature.

The simplest example (a mathematical object that Prof. Feller [23] said was the best all-purpose example in mathematics for understanding new concepts) is the unit disk along with its harmonic functions. The unit disk is the bounded complex domain, the unit circle is its Shilov boundary, and the harmonic functions are determined throughout the unit disk by their values on the Shilov boundary.

A more complicated example of such structures,
taken from the works of those mentioned above,
starts with an 8-real-dimensional 4-complex-dimensional
space denoted $\mathcal{M}^{\mathbf{C}}$ with signature $(2, 6)$.

What are Bounded Complex Homogeneous Domains?

To see the second example, start with the complexified Minkowski space-time $\mathcal{M}^{\mathbf{C}}_{2,6}$ of Penrose twistor theory.

$\mathcal{M}^{\mathbf{C}}$ is a Hermitian symmetric space that is the coset space

$$Spin(2,4)/Spin(1,3) \times U(1)$$

. In the mathematical classification notation, it is called a
Hermitian symmetric space of type $BDI_{2,4}$.

The Hermitian symmetric coset space is unbounded, but for each Hermitian symmetric space there exists a natural corresponding bounded complex domain.

In this case, $BCI_{2,4}$, the bounded complex domain is called $Type\ IV_4$ and consists of the elements of $\mathbf{C}_4$ defined by

$$\{z_1, z_2, z_3, z_4 \mid |z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 < (1 + |z_1^2 + z_2^2 + z_3^2 + z_4^2|)/2 < 1\} \quad (36)$$

What are Shilov boundaries?

The Shilov boundary (called the characteristic manifold by Hua in [40]) is a subset of the topological boundary of the bounded complex domain.

Following Hua [40], consider the analytic functions on the bounded complex domain. The Shilov boundary is the part of the topological boundary on which every analytic function attains its maximum modulus, and such that for every point on the Shilov boundary, there exists an analytic function on the bounded complex domain that attains its maximum modulus at that point.

The Shilov boundary is closed. Any function which is analytic in the neighborhood of every point of the Shilov boundary is uniquely determined by its values on the Shilov boundary.

In the case of the bounded domain of $Type\ IV_4$ Hua [40] shows that the Shilov boundary is

$$\{z = e^{i\theta}x \mid 0 \leq \theta \leq \pi, x\overline{x} = 1\} = \mathbf{R}P^1 \times S^3 \tag{37}$$

In the Penrose twistor formalism, it is the 4-dimensional Minkowski spacetime $\mathcal{M}_{1,3}$ with signature $(1, 3)$

There exists a kernel function, the Poisson kernel $P(z, \xi)$ function of a point z in the bounded complex domain and a point ξ in its Shilov boundary, such that, for any analytic $f(z)$,

$$f(z) = \int_{Shilovbdy} P(z, \xi) f(\xi) \tag{38}$$

Since all the analytic functions in the bounded complex domain are determinmed by their values on the Shilov boundary,the Shilov boundary should be the proper domain of definition of physically relevant functions.
That is why the $D_4 - D_5 - E_6$ model takes the Shilov boundary to be the relevant manifold for spacetime and for representing fermion particles and antiparticles.

There exists another kernel function,
the Bergman kernel function $K(z, \overline{w})$
of two points z, w in the bounded complex domain
such that, for any analytic $f(z)$,

$$f(z) = \int_{domain} K(z, \overline{w}) f(w) \tag{39}$$

Setting $z = w$ in the Bergman kernel gives a Riemannian metric for the bounded complex domain, which in turn defines invariant differential operators including the Laplacian, which in turn gives harmonic functions.

The Bergman kernel is equal to the ratio of the volume density to the Euclidean volume of the bounded complex domain.

Hua [40] not only gives the above description, he also actually calculates the volumes of the bounded complex domains and their Shilov boundaries.

Suppose that, as in the $D_4 - D_5 - E_6$ model, different bounded complex domains represent different physical forces whose Green's functions are determined by their invariant differential operators.
Then, since the domain volumes represent the measures of the Bergman kernels of bounded complex domains, and
since the physical part of the domain is its Shilov boundary, the ratios of (suitably normalized) volumes of Shilov boundaries should (and do, in the $D_4 - D_5 - E_6$ model) represent the ratios of force strengths of the corresponding forces.

4 $E_6/(D_5 \times U(1))$ and $D_5/(D_4 \times U(1))$.

4.1 2x2 Octonion Matrices and D5.

The 3×3 octonion traceless antihermitian checkerboard matrices form a subalgebra of the 3×3 octonion traceless antihermitian matrices.

$$\begin{pmatrix} S^7_1 & 0 & \mathbf{O}_v \\ 0 & S^7_2 & 0 \\ -\mathbf{O}^\dagger_v & 0 & -S^7_1 - S^7_2 \end{pmatrix} \tag{40}$$

It is isomorphic to the algebra of 2×2 octonion antihermitian matrices:

$$\begin{pmatrix} S^7_1 & \mathbf{O}_v \\ -\mathbf{O}^\dagger_v & S^7_2 \end{pmatrix} \tag{41}$$

When $Der(\mathbf{O}) = G_2$ is added the result is a subalgebra of the Lie algebra F_4:

$$\begin{pmatrix} S^7 & \mathbf{O}_v \\ -\mathbf{O}^\dagger_v & S^7_2 \end{pmatrix} \oplus G_2 \tag{42}$$

This Lie algebra is 8+7+7+14 = 36-dimensional $Spin(9)$, also denoted B_4. $Spin(9)$ is to $\mathbf{O}$ as $SU(2)$ is to $\mathbf{C}$.

The checkerboard 3×3 octonion traceless hermitian matrices also form a subalgebra of the 3×3 octonion traceless hermitian matrices:

$$\begin{pmatrix} a & 0 & \mathbf{O}_v \\ 0 & b & 0 \\ \mathbf{O}^\dagger_v & 0 & -a-b \end{pmatrix} \tag{43}$$

It is isomorphic to the 10-dimensional Jordan algebra $J_2^{\mathbf{O}}$ of 2×2 octonion hermitian matrices:

$$\begin{pmatrix} a & \mathbf{O}_v \\ \mathbf{O}^\dagger_v & b \end{pmatrix} \tag{44}$$

$J_2^{\mathbf{O}}$ has a 9-dimensional traceless Jordan subalgebra $J_2^{\mathbf{O}}o$, and

$$J_2^{\mathbf{O}} = J_2^{\mathbf{O}}o \oplus U(1)$$

The 1-dimensional Jordan algebra $J_1^{\mathbf{C}}o$ corresponds to the Lie algebra $U(1)$.

The 36-dimensional B_4 and the 9-dimensional $J_2^{\mathbf{O}}o$ combine to form the 45-dimensional Lie algebra D_5, also denoted $Spin(10)$, the D_5 of the $D_4 - D_5 - E_6$ model.

Physically, D_5 contains the gauge group and spacetime parts of E_6, so the half-spinor fermion particle and antiparticle parts of E_6 should live in a coset space that is a quotient of E_6 by D_5.

Due to complex structure, the quotient must be taken by $D_5 \times U(1)$ rather than D_5 alone. The $U(1)$ comes from action on the Jordan algebra $J_1^{\mathbf{C}}o$.

The resulting symmetric space that is the representation space of the first-generation particles and antiparticles in the $D_4 - D_5 - E_6$ model is the 78-45-1 = 32-real-dimensional hermitian symmetric space

$$E_6/(D_5 \times U(1)) \tag{45}$$

The 16 particles and antiparticles live on the 16-real-dimensional Shilov boundary of the bounded complex domain that corresponds to the symmetric space. The Shilov boundary is

$$(S^7 \times \mathbf{R}P^1) \oplus (S^7 \times \mathbf{R}P^1) \tag{46}$$

There is one copy of $S^7 \times \mathbf{R}P^1$ for the 8 first-generation fermion half-spinor particles, and one for the 8 antiparticles.

4.2 2x2 Diagonal Octonion Matrices and D4.

The 2×2 octonion antihermitian checkerboard matrices form the diagonal matrix subalgebra of the 2×2 octonion antihermitian matrices.

$$\begin{pmatrix} S^7_1 & 0 \\ 0 & S^7_2 \end{pmatrix} \tag{47}$$

When $Der(\mathbf{O}) = G_2$ is added the result is a subalgebra of the Lie algebras B_4 and F_4:

$$\begin{pmatrix} S^7_1 & 0 \\ 0 & S^7_2 \end{pmatrix} \oplus G_2 \tag{48}$$

This Lie algebra is 7+7+14 = 28-dimensional $Spin(0,8)$, also denoted D_4.

The checkerboard 2×2 octonion traceless hermitian matrices also form the diagonal subalgebra of the 2×2 octonion traceless hermitian matrices:

$$\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \tag{49}$$

It is isomorphic to the 1-dimensional Jordan algebra $J^{\mathbf{C}}_1 o$.

The 1-dimensional Jordan algebra $J^{\mathbf{C}}_1 o$ corresponds to the Lie algebra $U(1)$.

The 28-dimensional D_4 Lie algebra, also denoted $Spin(0,8)$, is (before dimensional reduction) the smallest Lie algebra in the $D_4 - D_5 - E_6$ model.

Physically, D_4 contains only the gauge group parts of D_5 and E_6, so the 8-dimensional spacetime of D_5 and E_6 should live in a coset space that is a quotient of D_5 by D_4.

Due to complex structure, the quotient must be taken by $D_4 \times U(1)$ rather than D_4 alone. The $U(1)$ comes from action on the Jordan algebra $J^{\mathbf{C}}_1 o$.

The resulting symmetric space that is the representation space of the 8-dimensional spacetime in the $D_4 - D_5 - E_6$ model is the 45-28-1 = 16-real-dimensional hermitian symmetric space

$$D_5/(D_4 \times U(1)) \tag{50}$$

The physical (before dimensional reduction) spacetime lives on the 8-real-dimensional Shilov boundary of the bounded complex domain that corresponds to the symmetric space. The Shilov boundary is

$$S^7 \times \mathbf{R}P^1 \tag{51}$$

5 Global E_6 Lagrangian.

Represent the E_6 Lie algebra by

$$\mathbf{R} \otimes \left(\begin{pmatrix} S_1^7 & \mathbf{O}_+ & \mathbf{O}_v \\ -\mathbf{O}_+^\dagger & S_2^7 & \mathbf{O}_- \\ -\mathbf{O}_v^\dagger & -\mathbf{O}_-^\dagger & -S_1^7 - S_2^7 \end{pmatrix} \oplus G_2 \oplus \begin{pmatrix} a & \mathbf{O}_+ & \mathbf{O}_v \\ \mathbf{O}_+^\dagger & b & \mathbf{O}_- \\ \mathbf{O}_v^\dagger & \mathbf{O}_-^\dagger & -a-b \end{pmatrix} \right) \quad (52)$$

where the $\mathbf{R}$ represents the real scalar field of this representation of E_6.

Full E_6 symmetry of the $D_4 - D_5 - E_6$ model is a global symmetry, and is useful primarily to:
define the spacetime, fermions, and bosons;
define the tree-level relative force strengths and particle masses;
give generalized supersymmetric relationships among fermions and bosons that may be useful in loop cancellations to produce ultraviolet and infrared finite results;
give CPT symmetry, which involves both particle C and spacetime P and T symmetries.
Some more discussion of CPT, CP, and T can be found at
WWW URL http://www.gatech.edu/tsmith/CPT.html [71].

Dynamics of the $D_4 - D_5 - E_6$ model are given by a Lagrangian action that is the integral over spacetime of a Lagrangian density made up of a gauge boson curvature term, a spinor fermion term (including through a Dirac operator interaction with gauge bosons), and a scalar term.

The 8-dimensional Lagrangian is:

$$\int_{V_8} F_8 \wedge \star F_8 + \partial_8^2 \overline{\Phi_8} \wedge \star \partial_8^2 \Phi_8 + \overline{S_{8\pm}} \not{\partial}_8 S_{8\pm} \quad (53)$$

where $\star$ is the Hodge dual;

∂_8 is the 8-dimensional covariant derivative,
$\not{\partial}_8$ is the 8-dimensional Dirac operator, and

F_8 is the 28-dimensional $Spin(8)$ curvature,
which come from the $Spin(0,8)$ gauge group subgroup of E_6, represented here by

$$\begin{pmatrix} S_1^7 & 0 \\ \\ 0 & S_2^7 \end{pmatrix} \oplus G_2 \tag{54}$$

Φ_8 is the 8-dimensional scalar field, which comes from the $\mathbf{R}$ scalar part of the representation of E_6;

V_8 is 8-dimensional spacetime, and which comes from the $\mathbf{O}_v$ part of the representation of E_6;

$S_{8\pm}$ are the $+$ and $-$ 8-dimensional half-spinor fermion spaces, which come from the $\mathbf{O}_+$ and $\mathbf{O}_-$ parts of the representation of E_6;

The 8-dimensional Lagrangian is a classical Lagrangian.

To get a quantum action, the $D_4 - D_5 - E6$ model uses a path integral sum over histories.
At its most fundamental level, the $D_4 - D_5 - E6$ model is a lattice model with 8-dimensional E_8 lattice spacetime.

The path integral sum over histories is based on a generalized Feynman checkerboard scheme over the E_8 lattice spacetime.

The original Feynman checkerboard is in 2-dimensional spacetime, in which the speed of light is naturally $c = 1$.

In the 4-dimensional spacetime of the D_4 lattice, the speed of light is naturally $c = \sqrt{3}$.

In 8-dimensional spacetime, the speed of light is naturally $c = \sqrt{7}$, but in the 8-dimensional E_8 lattice the nearest neighbor vertices have only 4 (not 8) non-zero coordinates and therefore have a natural speed of light of $c = \sqrt{3}$ that is appropriate for 4-dimensional light-cones rather than for 8-dimensional lightcones with $c = \sqrt{7}$.

This means that:

the generalized Feynman checkerboard scheme will not produce lightcone paths in 8-dimensional spacetime;

the dimension of spacetime must be reduced to 4 dimensions, as discussed in Section 7.; and
the 8-dimensional Lagrangian is classical, and does not contain gauge fixing and ghost terms.

Gauge-fixing term and ghost terms only appear after dimensional reduction of spacetime permits construction of generalized Feynman checkerboard quantum path integral sums over histories.

The gauge fixing terms are needed to avoid overcounting gauge-equivalent paths, and ghosts then appear.

The $D_4 = Spin(0,8)$ gauge symmetry will be reduced to gravity plus the Standard Model by dimensional reduction. The gauge symmetries of gravity and the Standard Model will be broken by gauge-fixing and ghost terms, and the resulting Lagrangian will have a BRS symmetry.

The dynamical symmetry that is physically relevant at the initial level is the $D_4 = Spin(0,8)$ gauge symmetry of the Lagrangian.

It is clear that $Spin(0,8)$ acts through its 8-dimensional vector, +half-spinor, and -half-spinor representations on the $\mathbf{O}_v$, $\mathbf{O}_+$, and $\mathbf{O}_-$ parts of the antihermitian matrix

$$\begin{pmatrix} S_1^7 & \mathbf{O}_+ & \mathbf{O}_v \\ -\mathbf{O}_+^\dagger & S_2^7 & \mathbf{O}_- \\ -\mathbf{O}_v^\dagger & -\mathbf{O}_-^\dagger & -S_1^7 - S_2^7 \end{pmatrix} \tag{55}$$

It is also clear that $Spin(0,8)$ acts through its 28-dimensional adjoint representation on the S_1^7 and S_2^7 parts of the antihermitian matrix, plus the derivations G_2, since

$$Spin(0,8) = S_1^7 \oplus S_2^7 \oplus G_2 \tag{56}$$

On the hermitian Jordan algebra matrix $J_3^{\mathbf{O}}o$:

$$\begin{pmatrix} a & \mathbf{O}_+ & \mathbf{O}_v \\ \mathbf{O}_+^\dagger & b & \mathbf{O}_- \\ \mathbf{O}_v^\dagger & \mathbf{O}_-^\dagger & -a - b \end{pmatrix} \tag{57}$$

the action of $Spin(0,8)$ (as a subgroup of E_6 and F_4) leaves invariant each of the $\mathbf{O}_v$, $\mathbf{O}_+$, and $\mathbf{O}_-$ parts of the matrix, corresponding to the facts that the gauge group $Spin(0,8)$:

does not act as a generalized supersymmetry to interchange spacetime and fermions; and

does not interchange fermion particles and antiparticles.

6 3 Generations: E_6, E_7, and E_8.

In the $D_4 - D_5 - E_6$ model, the first generation of spinor fermions is represented by the octonions $\mathbf{O}$, the second by $\mathbf{O} \oplus \mathbf{O}$, and the third by $\mathbf{O} \oplus \mathbf{O} \oplus \mathbf{O}$. The global structure of the $D_4 - D_5 - E_6$ model with 8-dimensional spacetime and first generation fermions is given by

$$E_6 = F_4 \oplus \begin{pmatrix} a & \mathbf{O}_+ & \mathbf{O}_v \\ \mathbf{O}_+^\dagger & b & \mathbf{O}_- \\ \mathbf{O}_v^\dagger & \mathbf{O}_-^\dagger & -a-b \end{pmatrix} \tag{58}$$

where

$$\begin{pmatrix} a & \mathbf{O}_+ & \mathbf{O}_v \\ \mathbf{O}_+^\dagger & b & \mathbf{O}_- \\ \mathbf{O}_v^\dagger & \mathbf{O}_-^\dagger & -a-b \end{pmatrix} = J_3^{\mathbf{O}} o \tag{59}$$

Here, the $\mathbf{O}_+$ and $\mathbf{O}$ in $J_3^{\mathbf{O}} o$ represent the $\mathbf{O}$ first generation fermion particles and antiparticles. Using the octonion basis $\{1, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ with quaternionic subalgebra basis $\{1, e_1, e_2, e_6\}$ and with octonion product Equation (4), the representation is:

Octonion basis element	*Fermion Particle*
1	*e − neutrino*
e_1	*red up quark*
e_2	*green up quark*
e_6	*blue up quark*
e_4	*electron*
e_3	*red down quark*
e_5	*green down quark*
e_7	*blue down quark*

(60)

The $\mathbf{O}_v$ represents 8-dimensional spacetime.

To represent the $\mathbf{O} \oplus \mathbf{O}$ second generation of fermions, you need a structure that generalizes Equation (52) by having two copies of the fermion octonions.

The simplest such generalization is

$$?E? = F_4 \oplus \begin{pmatrix} a & \mathbf{O}_+ & \mathbf{O}_v \\ \mathbf{O}_+^\dagger & b & \mathbf{O}_- \\ \mathbf{O}_v^\dagger & \mathbf{O}_-^\dagger & -a-b \end{pmatrix} \oplus \begin{pmatrix} 0 & \mathbf{O}_+ & 0 \\ \mathbf{O}_+^\dagger & 0 & \mathbf{O}_- \\ 0 & \mathbf{O}_-^\dagger & 0 \end{pmatrix} \qquad (61)$$

This proposal fails because the $\mathbf{O}_+$ and $\mathbf{O}$ in $?E?$ are not embedded in $J_3^{\mathbf{O}}o$ and therefore do not transform like the first generation $\mathbf{O}_+$ and $\mathbf{O}$ that are embedded in $J_3^{\mathbf{O}}o$.

Therefore, make the second simplest generalization:

$$??E?? = F_4 \oplus \begin{pmatrix} a & \mathbf{O}_+ & \mathbf{O}_v \\ \mathbf{O}_+^\dagger & b & \mathbf{O}_- \\ \mathbf{O}_v^\dagger & \mathbf{O}_-^\dagger & -a-b \end{pmatrix} \oplus \begin{pmatrix} a & \mathbf{O}_+ & \mathbf{O}_v \\ \mathbf{O}_+^\dagger & b & \mathbf{O}_- \\ \mathbf{O}_v^\dagger & \mathbf{O}_-^\dagger & -a-b \end{pmatrix} \qquad (62)$$

The $??E??$ proposal also fails, because the algebraic structure of the two copies of $J_3^{\mathbf{O}}o$ is incomplete.

To complete the algebraic structure , a third copy of $J_3^{\mathbf{O}}o$ must be added, and all three copies must be related algebraically like the imaginary quaternions $\{i, j, k\}$. This can be done by tensoring $J_3^{\mathbf{O}}o$ with the imaginary quaternions $S^3 = SU(2) = Spin(3) = Sp(1)$.

Since the order of the octonions in $\mathbf{O} \oplus \mathbf{O}$ should be irrelevant (for example, the octonion pair $\{e_i, 1\}$ should represent the same fermion as the octonion pair $\{1, e_i\}$), the structure must include the derivation algebra of the automorphism group of the quaternions, $SU(2)$.

The resulting structure is the 133-dimensional exceptional Lie algebra E_7:

$$E_7 = F_4 \oplus SU(2) \oplus S^3 \otimes \begin{pmatrix} a & \mathbf{O}_+ & \mathbf{O}_v \\ \mathbf{O}_+^\dagger & b & \mathbf{O}_- \\ \mathbf{O}_v^\dagger & \mathbf{O}_-^\dagger & -a-b \end{pmatrix} \tag{63}$$

Therefore E_7 is the global structure algebra of the second generation fermions.

For the third generation of fermions, note that three algebraically independent copies of $J_3^{\mathbf{O}}o$ generate seven copies, corresponding to the imaginary octonions $\{1, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$; that the imaginary octonions can be represented by S^7; and that the derivation algebra of the automorphism group of the octonions is G_2.

Therefore, the 248-dimensional exceptional Lie algebra E_8:

$$E_8 = F_4 \oplus G_2 \oplus S^7 \otimes \begin{pmatrix} a & \mathbf{O}_+ & \mathbf{O}_v \\ \mathbf{O}_+^\dagger & b & \mathbf{O}_- \\ \mathbf{O}_v^\dagger & \mathbf{O}_-^\dagger & -a-b \end{pmatrix} \tag{64}$$

is the global structure algebra of the third generation fermions.

Since there are only three Lie algebras in the series E_6, E_7, E_8, there are only three generations of fermions.

The first-generation Lie algebra E_6 has one copy of $J_3^{\mathbf{O}}o$, corresponding to the complex imaginary i.

The second-generation Lie algebra E_7 adds one more algebraically independent copy of $J_3^{\mathbf{O}}o$, corresponding to the quaternionic imaginary j.
Together with the i copy of $J_3^{\mathbf{O}}o$, the k copy is produced, so E_7 has in total 3 copies of $J_3^{\mathbf{O}}o$.

The third-generation Lie algebra E_8 adds one more algebraically independent copy of $J_3^{\mathbf{O}}o$, corresponding to the octonionic imaginary e_4.
Together with the $i = e_1, j = e_2, k = e_6$ copies of $J_3^{\mathbf{O}}o$, the e_3, e_5, e_7 copies are produced, so E_7 has in total 3 copies of $J_3^{\mathbf{O}}o$.

Therefore, transitions among generation of fermions to a lower one involve elimination of algebraically independent imaginaries.

From 2nd to 1st, it must effectively map $j \rightarrow 1$.
This transition can be done in one step.

From 3rd to 2nd, it must effectively map $e_4 \rightarrow 1$. This transition can also be done in one step.

From 3rd to 1st, it must effectively map $j, e_4 \rightarrow 1$. This transition cannot be done in one step.
The map $e_4 \rightarrow 1$ only gets you to the 2nd generation.
You still need the additional map $j \rightarrow 1$ to get to the 1st.

Since the imaginaries e_4 and j are orthogonal to each other, there must be an intermediate step that is effectively a phase shift of $\pi/2$, and the KOBAYASHI-MASKAWA PHASE angle parameter should be

$$\epsilon = \pi/2 \tag{65}$$

7 Discrete Lattices and Dimensional Reduction.

The physical manifolds of the $D_4 - D_5 - E_6$ model should be representable in terms of discrete lattices in order to be formulated as a generalized Feynman checkerboard model.

General references on lattices, polytopes, and related structures are the book of Conway and Sloane [9] and the books [10, 11] and some papers [12, 13] of Coxeter.

7.1 Discrete Lattice $D_4 - D_5 - E_6$ Model.

As seen in section 2, 78-dimensional E_6 of the $D_4 - D_5 - E_6$ model is made up of three parts:
38-dimensional space of antihermitian 3×3 octonion matrices;
14-dimensional space of the Lie algebra G_2; and
26-dimensional space of the Jordan algebra $J_3^{\mathbf{O}}o$.

7.2 Antihermitian 3×3 Octonion Matrices.

Antihermitian 3×3 octonion matrices:

$$\begin{pmatrix} S_1^7 & \mathbf{O}_+ & \mathbf{O}_v \\ -\mathbf{O}_+^\dagger & S_2^7 & \mathbf{O}_- \\ -\mathbf{O}_v^\dagger & -\mathbf{O}_-^\dagger & -S_1^7 - S_2^7 \end{pmatrix} \tag{66}$$

Individually, $\mathbf{O}_v$, $\mathbf{O}_+$, and $\mathbf{O}_-$ can each be represented by the 8-dimensional E_8 lattice, as shown by Geoffrey Dixon [20].

As Geoffrey Dixon [21] has shown, the two half-spinor spaces $\mathbf{O}_+$, and $\mathbf{O}_-$ taken together can be represented by the 16-dimensional Barnes-Wall lattice Λ_{16}.

The Barnes-Wall lattices form a series of real dimension 2^n for $n \geq 2$, and that the Barnes-Wall lattices of real dimension 4 and 8 are the D_4 and E_8 lattices.

$\mathbf{O}_+$, and $\mathbf{O}_-$ form the Shilov boundary of a 16-complex dimensional bounded complex homogeneous domain,
and Conway and Sloane [9] note that
Quebbemann's 32-real dimensional lattice is a complex lattice whose 16-real-dimensional real part is the 16-real-dimensional Barnes-Wall Λ_{16} lattice.

To represent all three $\mathbf{O}_v$, $\mathbf{O}_+$, and $\mathbf{O}_-$ together, it may be possible to use the 24-dimensional Leech lattice.

Such a Leech lattice representation is likely to be related to the group

$$Spin(0,8) = S^7 \bowtie S^7 \bowtie G_2$$

where $\bowtie$ denotes the fibre product of the fibrations

$$Spin(7) \rightarrow Spin(8) \rightarrow S^7$$

and

$$G_2 \rightarrow Spin(7) \rightarrow S^7$$

In a Leech lattice representation, it is likely that:

the $\mathbf{O}_+$ and $\mathbf{O}_-$ correspond to the two S^7's of the $Spin(0,8)$ fibrations, and to the X-product of Martin Cederwall [6] and the XY-product on which Geoffrey Dixon is working; and

the $\mathbf{O}_v$ corresponds to a 7-dimensional representation of the G_2 of the $Spin(0,8)$ fibrations.

The entire 26-dimensional space of traceless antihermitian 3×3 octonion matrices may be represented by the Lorentz Leech lattice $\Pi_{25,1}$,
which is closely related to the Monster group.

7.3 $S^7 \oplus S^7 \oplus G_2 = D_4$.

$S^7_1 \oplus S^7_2 \oplus G_2 = D_4$ is the Lie algebra of the $Spin(0,8)$ gauge group of the $D_4 - D_5 - E_6$ model (before spacetime dimensional reduction).

The gauge group $Spin(0,8)$ acts through its gauge bosons, which can:
propagate through spacetime;
interact with fermion particles or antiparticles; and
interact with each other.

Propagation through spacetime is represented by the 8-dimensional vector representation of $Spin(0,8)$, which in turn is represented by the octonions $\mathbf{O}_v$ and, in discrete lattice version, by the E_8 lattice.

Interaction with fermion particles is represented by the 8-dimensional +half-spinor representation of $Spin(0,8)$, which in turn is represented by the octonions $\mathbf{O}_+$ and, in discrete lattice version, individually by the E_8 lattice.

Interaction with fermion antiparticles is represented by the 8-dimensional -half-spinor representation of $Spin(0,8)$, which in turn is represented by the octonions $\mathbf{O}_-$ and, in discrete lattice version, individually by the E_8 lattice.

Together, the two half-spinor representations are
represented by the Barnes-Wall Λ_{16} lattice.

Interaction with other gauge bosons is represented by the 28-dimensional adjoint representation of $Spin(0,8)$, which in turn is represented by the exterior wedge bivector product of two copies of the vector octonions $\mathbf{O}_v \wedge \mathbf{O}_v$ and, in discrete lattice version, by the exterior wedge bivector product of two copies of the E_8 lattice, $E_8 \wedge E_8$.

7.4 Jordan algebra $J_3^{\mathbf{O}}o$.

26-dimensional space of the Jordan algebra $J_3^{\mathbf{O}}o$:

$$\begin{pmatrix} a & \mathbf{O}_+ & \mathbf{O}_v \\ \mathbf{O}_+^\dagger & b & \mathbf{O}_- \\ \mathbf{O}_v^\dagger & \mathbf{O}_-^\dagger & -a-b \end{pmatrix} \tag{67}$$

The 25+1 = 26-dimensional Lorentz Leech lattice $II_{25,1}$ can be used to represent the 26-dimensional Jordan algebra $J_3^{\mathbf{O}}o$.

As seen in section 4.,the physically relevant group action on $J_3^{\mathbf{O}}o$ for the Lagrangian dynamics of the $D_4 - D_5 - E_6$ model is not action by the global symmetry group E_6, but rather action by the gauge group $Spin(0,8)$.

Since action on $J_3^{\mathbf{O}}o$ by the gauge group $Spin(0,8)$ leaves invariant each of the $\mathbf{O}_v$, $\mathbf{O}_+$, and $\mathbf{O}_-$ parts of the matrix, the only discrete lattice structure needed for the Lagrangian dynamics of the $D_4 - D_5 - E_6$ model is the representation of each of $\mathbf{O}_v$, $\mathbf{O}_+$, and $\mathbf{O}_-$ by an E_8 lattice, and the full Lorentz Leech lattice $II_{25,1}$ is not needed.

However, in case it may be useful to have a discrete lattice description of global symmetries (for example, in looking at generalized supersymmetric relationships among fermions and bosons or at CPT symmetry), here are some characteristics of the Lorentz Leech lattice $II_{25,1}$:

$II_{25,1}$ can be represented (as in Conway and Sloane [9]) by the set of vectors $\{x_0, x_1, ..., x_{24}|x_{25}\}$ such that all the x_i are in $\mathbf{Z}$ or all in $\mathbf{Z} + 1/2$ and satisfy
$x_0 + ... + x_{24} - x_{25} \in 2\mathbf{Z}$

Let $w = (0, 1, 2, 3, ...23, 24|70)$. Since $0^2 + 1^2 + 2^2 + ... + 24^2 = 70^2$, w is an isotropic vector in $II_{25,1}$.

Then the Leech roots, or vectors r in $II_{25,1}$ such that $r \cdot r = 2$ and $r \cdot w = -1$ are the vertices of a 24-dimensional Leech lattice.

Also, $(w^{\perp} \cap II_{25,1})/w$ is a copy of the Leech lattice.

The 24-dimensional Leech lattice can be made up of 3 E_8 lattices, and so corresponds to the off-diagonal part of $J_3^{\mathbf{O}}o$.

Each vertex of the Leech lattice has 196,560 nearest neighbors.

A space of 196,560+300+24 = 196,884 dimensions can be used to represent the largest finite simple, group, the Fischer-Greiss Monster of order

$$2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 =$$

$$= 808, 017, 424, 794, 512, 875, 886, 459, 904,$$
$$961, 710, 757, 005, 754, 368, 000, 000, 000$$

7.5 Lattice Dimensional Reduction.

7.5.1 HyperDiamond Lattices.

The lattices of type D_n are n-dimensional checkerboard lattices, that is, the alternate vertices of a Zn hypercubic lattice. A general reference on lattices is Conway and Sloane [9]. For the n-dimensional HyperDiamond construction from D_n, Conway and Sloane use an n-dimensional glue vector $[1] = (0.5, ..., 0.5)$ (with n 0.5's).

Consider D_3, the fcc close packing in 3-space. Make a second D_3 shifted by the glue vector $(0.5, 0.5, 0.5)$.

Then form the union $D_3 \cup ([1] + D_3)$.

That is a 3-dimensional Diamond crystal.

When you do the same thing to get a 4-dimensional HyperDiamond, you get $D_8 \cup ([1] + D_8)$.
The 4-dimensional HyperDiamond $D_4 \cup ([1]+D_4)$ is the $\mathbf{Z}^4$ hypercubic lattice with null edges.

It is the lattice that Michael Gibbs [30] uses in his Ph. D. thesis advised by David Finkelstein.

When you construct an 8-dimensional HyperDiamond, you get $D_8 \cup ([1] + D_8) = E_8$, the fundamental lattice of the octonion structures in the $D_4 - D_5 - E_6$ model described in hep-ph/9501252.

7.5.2 Dimensional Reduction.

Dimensional reduction of spacetime from 8 to 4 dimensions takes the E_8 lattice into a D_4 lattice.
The E_8 lattice can be written in 7 different ways using
octonion coordinates with basis

$$\{1, e_1, e_2, e_3, e_4, e_5, e_6, e_7\} \tag{68}$$

One way is:
16 vertices:

$$\pm 1, \pm e_1, \pm e_2, \pm e_3, \pm e_4, \pm e_5, \pm e_6, \pm e_7 \tag{69}$$

96 vertices:

$$\begin{array}{c}(\pm 1 \pm e_1 \pm e_2 \pm e_3)/2 \\ (\pm 1 \pm e_2 \pm e_5 \pm e_7)/2 \\ (\pm 1 \pm e_2 \pm e_4 \pm e_6)/2 \\ (\pm e_4 \pm e_5 \pm e_6 \pm e_7)/2 \\ (\pm e_1 \pm e_3 \pm e_4 \pm e_6)/2 \\ (\pm e_1 \pm e_3 \pm e_5 \pm e_7)/2\end{array} \tag{70}$$

128 vertices:

$$\begin{array}{c}(\pm 1 \pm e_3 \pm e_4 \pm e_7)/2 \\ (\pm 1 \pm e_1 \pm e_5 \pm e_6)/2 \\ (\pm 1 \pm e_3 \pm e_6 \pm e_7)/2 \\ (\pm 1 \pm e_1 \pm e_4 \pm e_7)/2 \\ (\pm e_1 \pm e_2 \pm e_6 \pm e_7)/2 \\ (\pm e_2 \pm e_3 \pm e_4 \pm e_7)/2 \\ (\pm e_1 \pm e_2 \pm e_4 \pm e_5)/2 \\ (\pm e_2 \pm e_3 \pm e_5 \pm e_6)/2\end{array} \tag{71}$$

Consider the quaternionic subspace of the octonions
with basis $\{1, e_1, e_2, e_6\}$ and
the D_4 lattice with origin nearest neighbors:
8 vertices:

$$\pm 1, \pm e_1, \pm e_2, \pm e_6 \tag{72}$$

and
16 vertices:

$$(\pm 1 \pm e_1 \pm e_2 \pm e_6)/2 \tag{73}$$

Dimensional reduction of the E_8 lattice spacetime to 4-dimensional spacetime reduces each of the D_8 lattices in the E_8 lattice to D_4 lattices.

Therefore, we should get a 4-dimensional HyperDiamond $D_4 \cup ([1] + D_4)$.

The 4-dimensional HyperDiamond $D_4 \cup ([1] + D_4)$ is the $\mathbf{Z}^4$ hypercubic lattice with null edges.

It is the lattice that Michael Gibbs [30] uses in his Ph. D. thesis advised by David Finkelstein.

Here is an explicit construction of the 4-dimensional HyperDiamond.

START WITH THE 24 VERTICES OF A 24-CELL D_4:

$$\begin{array}{cccc}
+1 & +1 & 0 & 0 \\
+1 & 0 & +1 & 0 \\
+1 & 0 & 0 & +1 \\
+1 & -1 & 0 & 0 \\
+1 & 0 & -1 & 0 \\
+1 & 0 & 0 & -1 \\
-1 & +1 & 0 & 0 \\
-1 & 0 & +1 & 0 \\
-1 & 0 & 0 & +1 \\
-1 & -1 & 0 & 0 \\
-1 & 0 & -1 & 0 \\
-1 & 0 & 0 & -1 \\
0 & +1 & +1 & 0 \\
0 & +1 & 0 & +1 \\
0 & +1 & -1 & 0 \\
0 & +1 & 0 & -1 \\
0 & -1 & +1 & 0 \\
0 & -1 & 0 & +1 \\
0 & -1 & -1 & 0 \\
0 & -1 & 0 & -1 \\
0 & 0 & +1 & +1 \\
0 & 0 & +1 & -1 \\
0 & 0 & -1 & +1 \\
0 & 0 & -1 & -1
\end{array} \tag{74}$$

SHIFT THE LATTICE BY A GLUE VECTOR, BY ADDING

$$0.5 \quad 0.5 \quad 0.5 \quad 0.5 \tag{75}$$

TO GET 24 MORE VERTICES $[1] + D_4$:

$$\begin{array}{cccc}
+1.5 & +1.5 & 0.5 & 0.5 \\
+1.5 & 0.5 & +1.5 & 0.5 \\
+1.5 & 0.5 & 0.5 & +1.5 \\
+1.5 & -0.5 & 0.5 & 0.5 \\
+1.5 & 0.5 & -0.5 & 0.5 \\
+1.5 & 0.5 & 0.5 & -0.5 \\
-0.5 & +1.5 & 0.5 & 0.5 \\
-0.5 & 0.5 & +1.5 & 0.5 \\
-0.5 & 0.5 & 0.5 & +1.5 \\
-0.5 & -0.5 & 0.5 & 0.5 \\
-0.5 & 0.5 & -0.5 & 0.5 \\
-0.5 & 0.5 & 0.5 & -0.5 \\
0.5 & +1.5 & +1.5 & 0.5 \\
0.5 & +1.5 & 0.5 & +1.5 \\
0.5 & +1.5 & -0.5 & 0.5 \\
0.5 & +1.5 & 0.5 & -0.5 \\
0.5 & -0.5 & +1.5 & 0.5 \\
0.5 & -0.5 & 0.5 & +1.5 \\
0.5 & -0.5 & -0.5 & 0.5 \\
0.5 & -0.5 & 0.5 & -0.5 \\
0.5 & 0.5 & +1.5 & +1.5 \\
0.5 & 0.5 & +1.5 & -0.5 \\
0.5 & 0.5 & -0.5 & +1.5 \\
0.5 & 0.5 & -0.5 & -0.5
\end{array} \tag{76}$$

FOR THE NEW COMBINED LATTICE $D_4 \cup ([1] + D_4)$, THESE ARE 6 OF THE NEAREST NEIGHBORS TO THE ORIGIN:

$$\begin{array}{cccc} -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & 0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & -0.5 & 0.5 \\ 0.5 & -0.5 & 0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \end{array} \tag{77}$$

HERE ARE 2 MORE THAT COME FROM ADDING THE GLUE VECTOR TO LATTICE VECTORS THAT ARE NOT NEAREST NEIGHBORS OF THE ORIGIN:

$$\begin{array}{cccc} 0.5 & 0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & -0.5 & -0.5 \end{array} \tag{78}$$

THEY COME, RESPECTIVELY, FROM ADDING THE GLUE VECTOR TO:

THE ORIGIN

$$0 \quad 0 \quad 0 \quad 0 \tag{79}$$

ITSELF;

AND

THE LATTICE POINT

$$-1 \quad -1 \quad -1 \quad -1 \tag{80}$$

WHICH IS SECOND ORDER, FROM

$$\begin{array}{cccc} -1 & -1 & 0 & 0 \\ plus & & & \\ 0 & 0 & 0 & 0 \end{array} \tag{81}$$

FROM

$$\begin{array}{cccc} -1 & 0 & -1 & 0 \\ plus & & & \\ 0 & -1 & 0 & -1 \end{array} \tag{82}$$

OR FROM

$$\begin{array}{cccc} -1 & 0 & 0 & -1 \\ plus & & & \\ 0 & -1 & -1 & 0 \end{array} \tag{83}$$

That the E_8 lattice is, in a sense, fundamentally 4-dimensional can be seen from several points of view:

the E_8 lattice nearest neighbor vertices have only 4 non-zero coordinates, like 4-dimensional spacetime with speed of light $c = \sqrt{3}$, rather than 8 non-zero coordinates, like 8-dimensional spacetime with speed of light $c = \sqrt{7}$, so the E_8 lattice light-cone structure appears to be 4-dimensional rather than 8-dimensional;

the representation of the E_8 lattice by quaternionic icosians, as described by Conway and Sloane [9];

the Golden ratio construction of the E_8 lattice from the D_4 lattice, which has a 24-cell nearest neighbor polytope (The construction starts with the 24 vertices of a 24-cell, then adds Golden ratio points on each of the 96 edges of the 24-cell, then extends the space to 8 dimensions by considering the algebraicaly independent $\sqrt{5}$ part of the coordinates to be geometrically independent, and finally doubling the resulting 120 vertices in 8-dimensional space (by considering both the D_4 lattice and its dual D_4^*) to get the 240 vertices of the E_8 lattice nearest neighbor polytope (the Witting polytope); and

the fact that the 240-vertex Witting polytope, the E_8 lattice nearest neighbor polytope, most naturally lives in 4 complex dimensions, where it is self-dual, rather than in 8 real dimensions.

Some more material on such things can be found at
WWW URL http://www.gatech.edu/tsmith/home.html [71].

In referring to Conway and Sloane [9], bear in mind that they use the convention (usual in working with lattices) that the norm of a lattice distance is the square of the length of the lattice distance.

It is also noteworthy that the number of vertices in shells of an E_8 lattice increase monotonically as the radius of the shell increases,
while cyclic relationships (see Conway and Sloane [9]) appear in the number of vertices in shells of a D_4 lattice.

7.6 Discrete Lattice Effects of $8 \to 4$ *dim.*

$Spin(0,8)$ acts on the octonions **O**, the lattice version of which is the E_8 lattice.

Each vertex in the E_8 lattice has 240 nearest neighbors, the inner shell of the E_8 lattice.

Geoffrey Dixon [21] shows that the 240 vertices in the E_8 inner shell break down with respect to the two 4-dimensional subspaces of **O**, each represented by the inner shell of a D_4 lattice, as

$$\begin{array}{rcl} <U,0> & \to & 24\ elements \\ <0,V> & \to & 24\ elements \\ <W,X>(WX* = +/-qm) & \to & 192\ elements \end{array} \tag{84}$$

where D_4^* is the dual lattice to the D_4 lattice, and where $U,V \in D_4$, $W \in D_4^*$, and $X \in \{\pm 1, \pm i, \pm j, \pm k\} \subset D_4^*$

The two 24-element sets each have the group structure of the binary tetrahedral group, also the group of 24 quaternion units, and the 24 elements would represent the root vectors of the $Spin0,(8)$ D_4 Lie algebra in the 4-dimensional space of the D_4 lattice.

The 192 element set is the Weyl group of the $Spin0,(8)$ D_4 Lie algebra. The Weyl group is the group of reflections in the hyperplanes (in the D_4 4-dimensional space) that are orthogonal to the 24 root vectors.

If the 8-dimensional E_8 spacetime is reduced to the 4-dimensional D_4 spacetime, then

$$\begin{array}{rcc} <U,0> & \to & 24\ elements\ of\ D_4\ lattice \\ & & inner\ shell \\ <0,V> & \to & 24\ elements\ of\ binary \\ & & tetrahedral\ group \\ <W,X>(WX* = +/-qm) & \to & 192\ elements\ of\ Weyl\ group \\ & & of\ reduced\ gauge\ group \end{array} \tag{85}$$

where $U,V \in D_4$, $W \in D_4^*$, and $X \in \{\pm 1, \pm i, \pm j, \pm k\} \subset D_4^*$

The 24-element D_4 lattice inner shell formed by the U elements of Equation (76) form the second shell of the reduced 4-dimensional spacetime D_4 lattice, as described in the preceding Subsection 7.5.
Denote this set by $24U$.

The 24-element D_4 lattice inner shell formed by the V elements of Equation (76) form the 24-element finite group (binary tetrahedral group of unit quaternions) that is the Weyl group of the internal symmetry gauge group of the 4-dimensional $D_4 - D_5 - E_6$ model.
Denote this group by $24V$.

The 192-element Weyl group of the D_4 Lie algebra of $Spin(0,8)$ is made up of pairs, the first of which is an element of the 24-element set of W elements.
Denote that 24-element set by $24W$.
The second part of a pair is an element of the 8-element set of X elements.
Denote that 8-element set by $8X$.

After dimensional reduction of spacetime, $Spin(0,8)$ is too big to act as an isotropy group on spacetime, as it acted in 8-dimensional spacetime, which is of the form

$$Spin(2,8)/Spin(2,0) \times Spin(0,8)$$

.

Therefore, in 4-dimensional spacetime, the 28 infinitesimal generators of $Spin(0,8)$ (which act as 28 gauge bosons in 8-dimensional $D_4 - D_5 - E_6$ physics) cannot interact according to the commutation relations of $Spin(0,8)$, but must interact according to commutation relations of smaller groups that can act on 4-dimensional spacetime.

What type of action should these smaller groups have on 4-dimensional spacetime?

Isotropy action is sufficient in the case of the 28 $Spin(0,8)$ infinitesimal generators in the 8-dimensional theory, because the gauge boson part of the Lagrangian $\int_{8-dim} F \wedge \star F$ is an integral over 8-dimensional spacetime

using a uniform measure that is the same for all 28 gauge boson infinitesimal generators. Physically, the force strength of the $Spin(0,8)$ gauge group can be taken to be 1 because there is only one gauge group in the 8-dimensional $D_4 - D_5 - E_6$ model.
However, as we shall see now, isotropy action is not sufficient for the gauge groups after dimensional reduction.

After reduction to 4-dimensional spacetime, the 28 infinitesimal generators will have to regroup into more that one smaller groups, each of which will have its own force strength. The 4-dimensional Lagrangian will be the sum of more than one Lagrangians of the form $\int_{4-dim} F \wedge \star F$, each of which will use a different measure in integrating over 4-dimensional spacetime.
A factor in determining the relative strengths of the 4-dimensional forces will be the relative magnitude of the measures over 4-dimensional spacetime. Therefore, the measure information should be carried in the F of the 4-dimensional Lagrangian $\int_{4-dim} F \wedge \star F$, which is different for each force, rather than in the overall $\int_{4-dim}$, which should be uniform for all terms in the total 4-dimensional Lagrangian of the $D_4 - D_5 - E_6$ model.

In the 4-dimensional $D_4 - D_5 - E_6$ model, the small gauge groups must act transitively on 4-dimensional spacetime, so that they can carry the measure information. Physically, the gauge bosons of the different gauge groups see spacetime differently
(see WWW URL http://www.gatech.edu/tsmith/See.html [71]).

Since the 4-dimensional $D_4 - D_5 - E_6$ model gauge groups
are not isotropy groups of 4-dimensional spacetime,
but actually act transitively on 4-dimensional manifolds,
they are not exactly conventional "local symmetry" gauge groups.
In the conventional "local symmetry" picture, you can put a gauge boson infinitesimal generator $x(G)$ of the gauge group G at each point p of the spacetime base manifold, with the choice of $x(G)$ made independently at each point p.
In the $D_4 - D_5 - E_6$ model picture, the choice of $x(G)$ at a point not only is the choice of a gauge boson, but also of a "translation" direction in the 4-dimensional spacetime.

When you choose a gauge boson, say a 'red-antiblue gluon" of the $SU(3)$ color force gauge group, then does your "choice" of "translation direction" fix a physical direction of propagation (say, $(+1 - i - j - k)/2$ in quaternionic coordinates for the future lightcone) ?

If it does, the model doesn't work right. Fortunately for the $D_4 - D_5 - E_6$ model, there is one more choice to be made independently at each point, and that is the choice to put any given element of the gauge group G in corresponce with with any element of the isotropy subgroup K or with any direction in the 4-dimensional manifold G/K on which G acts transitively.

Therefore, at any given point in the 4-dimensional spacetime, you can choose the "red-antiblue gluon" gauge boson (or any other gauge boson) and the $(+1 - i - j - k)/2$ direction (or any other direction), and independent choices can be made at all points in the 4-dimensional spacetime of the $D_4 - D_5 - E_6$ model.

Since choices of gauge boson and direction of propagation are both made at once, it is natural in the lattice $D_4 - D_5 - E_6$ model to picture the gauge bosons as living on the links of the spacetime lattice, with the fermion particles and antiparticles living on the vertices.

This type of structure might not work consistently in a model with less symmetry than the $D_4 - D_5 - E_6$ model. Particularly, the triality symmetry of 8-dimensional spacetime with the 8-dimensional half-spinor representation spaces of the first-generation fermion particles and antiparticles means that the gauge boson symmetry group, which must act on fermion particles and antiparticles in a natural way,

also acts transitively on spacetime in a natural way.
For a discussion of what types of "generalized supersymmetry" symmetries are useful, and why the $D_4 - D_5 - E_6$ model probably has the most useful symmetries of any model, see hep-th/9306011 [68]).

Even after dimensional reduction of spacetime, there is a residual symmetry relationship between the fermion representation spaces and spacetime. Perhaps that residual symmetry might be a way to relate the results of the $D_4 - D_5 - E_6$ model to the results of the 4-dimensional lattice model of Finkelstein and Gibbs. [30]

What can these smaller groups be? Since 4-dimensional spacetime has quaternionic structure, they must act transitively on 4-dimensional manifolds with quaternionic structure.
Such manifolds have been classified by Wolf [76], and they are

M	$Symmetric\,Space$	$Gauge\,Group$
S^4	$\frac{Spin(5)}{Spin(4)}$	$Spin(5)$
$\mathbf{C}P^2$	$\frac{SU(3)}{SU(2)\times U(1)}$	$SU(3)$
$S^2 \times S^2$	$2\ copies\ of\ \left(\frac{SU(2)}{U(1)}\right)$	$SU(2)$
$S^1 \times S^1 \times S^1 \times S^1$	$4\ copies\ of\ U(1)$	$U(1)$

(86)

Therefore the 4 forces have gauge groups
$Spin(5)$ (10 infinitesimal generators)
$SU(3)$ (8 infinitesimal generators)
$SU(2)$ (2 copies, each with 3 infinitesimal generators)
and
$U(1)$ (4 copies, each with 1 infinitesimal generator)
that account for all 28 of the $Spin(0,8)$ gauge bosons.

There are two cases in which 4-dimensional spacetime is made up of multiple copies of lower-dimensional manifolds.

Two copies of the gauge group $SU(2)$ act on 2 copies of S^2.
Each $SU(2)$ has 3 infinitesimal generators,
the three weak bosons $\{W_-, W_0, W_+$.
Since a given weak boson cannot carry a different charge in
different parts of 4-dimensional spacetime,
the 2 copies of $SU(2)$ must be aligned consistently.
This means that there is physically only one $SU(2)$ weak force gauge group,
and that there are 3 degrees of freedom due to
3 $Spin(0, 8)$ infinitesimal generators that are not used.
Denote them by $3 - SU(2)$.

Four copies of the gauge group $U(1)$ act on 4 copies of S^1.
Each $U(1)$ has 1 infinitesimal generator, the photon.
Since a given photon should be the same in all parts of 4-dimensional spacetime,
the 4 copies of $U(1)$ must be aligned consistently.
This means that there is physically only one $U(1)$ electromagnetic gauge group,
and that there are 3 degrees of freedom due to
3 $Spin(0, 8)$ infinitesimal generators that are not used.
Denote them by $3 - U(1)$.

How does all this fit into the structures $24U$, $24V$, $24W$, and $8X$?

$24U$ and $24W$ are the D_4 and D_4^* of the 4-dimensional lattice spacetime as described in the preceding Subsection 7.5 .

The $8X$, being part of a 192-element product with $24W$, should represent a gauge group that is closely connected to spacetime. $8X$ represents the 8-element Weyl group S_2^3 of the gauge group $Spin(5)$.
By the MacDowell-Mansouri mechanism [44], the $Spin(5)$ gauge group accounts for Einstein-Hilbert gravity.

$24V$, being entirely from the part of 8-dimensional spacetime that did not survive dimensional reduction, should represent the Weyl groups of gauge groups of internal symmetries.

As $24V$ is the 24-element binary tetrahedral group of unit quaternions, the inner shell of the D_4 lattice, it has a 4-element subgroup S_2^2 made up of unit complex numbers, the inner shell of the Gaussian lattice.
$24V/S_2^2$ is the 6-element group S_3, which is the Weyl group of the $SU(3)$ gauge group of the color force.
The S_2^2 is 2 copies of the Weyl group of the $SU(2)$ gauge group of the weak force.
As $U(1)$ is Abelian, and so has the identity for its Weyl group, 4 copies of the $U(1)$ gauge group of electromagnetism can be said to be included in the gauge groups of which $24V$ is the Weyl group.

Therefore, $24V$ gives us the gauge groups of the Standard Model, $SU(3) \times SU(2) \times U(1)$,
plus the extra 6 degrees of freedom $3-SU(2)$ and $3-U(1)$ that were discussed above.

It will be seen in Section 8.2 that 5 of the 6 extra degrees of freedom are a link between the Higgs sector of the Standard Model and conformal symmetry related to gravity, and the 6th, a copy of U(1), accounts for the complex phase of propagators in the $D_4 - D_5 - E_6$ model.

From the continuum limit viewpoint of Section 8.2,
those 6 degrees of freedom are combined with the gravity sector
to expand the 10-dimensional de Sitter $Spin(5)$ group
of the MacDowell-Mansouri mechanism
to the 15-dimensional conformal $Spin(2,4)$ group
plus the $U(1)$ of the complex propagator phase,
which in turn can all be combined into
one copy of 16-dimensional $U(4)$.

From the discrete Weyl group point of view of this section, the $3-SU(2)$ and $3-U(1)$ degrees of freedom are represented by the Weyl group S_2.
The S_2 can be identified with the 3 S_2's of the gravity $Spin(5)$ S_2^3 in 3 ways, so the identification effectively expands the gravity sector Weyl group by a factor of 3 from $S_2^3 = S_2^2 \times S_2$ to $S_2^2 \times S_3$, which is the Weyl group of the conformal group $Spin(2,4)$ and also the Weyl group of $U(4)$.

8 Continuum Limit Effects of $8 \rightarrow 4\, dim$.

The dimensional reduction breaks the gauge group $Spin(0,8)$ into gravity plus the Standard Model.

The following sections discuss the effects of dimensional reduction on the terms of the 8-dimensional Lagrangian

$$\int_{V_8} F_8 \wedge \star F_8 + \partial_8^2 \overline{\Phi_8} \wedge \star \partial_8^2 \Phi_8 + \overline{S_{8\pm}}\ \not{\partial}_8 S_{8\pm} \tag{87}$$

of the $D_4 - D_5 - E_6$ model discussed in Section 5.

and the phenomenological results of calculations based on the 4-dimensional structures.

8.1 Scalar part of the Lagrangian

The scalar part of the 8-dimensional Lagrangian is

$$\int_{V_8} \partial_8^2 \overline{\Phi_8} \wedge \star \partial_8^2 \Phi_8$$

As shown in chapter 4 of Göckeler and Schücker [33], $\partial_8^2 \Phi_8$ can be represented as an 8-dimensional curvature F_{H8}, giving

$$\int_{V_8} F_{H8} \wedge \star F_{H8}$$

When spacetime is reduced to 4 dimensions, denote the surviving 4 dimensions by 4 and the reduced 4 dimensions by $\perp 4$.

Then, $F_{H8} = F_{H44} + F_{H4\perp 4} + F_{H\perp 4\perp 4}$, where
F_{H44} is the part of F_{H8} entirely in the surviving spacetime;
$F_{H4\perp 4}$ is the part of F_{H8} partly in the surviving spacetime and partly in the reduced spacetime; and
$F_{H\perp 4\perp 4}$ is the part of F_{H8} entirely in the reduced spacetime;

The 4-dimensional Higgs Lagrangian is then:
$\int (F_{H44} + F_{H4\perp 4} + F_{H\perp 4\perp 4}) \wedge \star (F_{H44} + F_{H4\perp 4} + F_{H\perp 4\perp 4}) =$
$= \int (F_{H44} \wedge \star F_{H44} + F_{H4\perp 4} \wedge \star F_{H4\perp 4} + F_{H\perp 4\perp 4} \wedge \star F_{H\perp 4\perp 4})$.

As all possible paths should be taken into account in the sum over histories path integral picture of quantum field theory, the terms involving the reduced 4 dimensions, $\perp 4$, should be integrated over the reduced 4 dimensions.

Integrating over the reduced 4 dimensions, $\perp 4$, gives
$\int (F_{H44} \wedge \star F_{H44} + \int_{\perp 4} F_{H4\perp 4} \wedge \star F_{H4\perp 4} + \int_{\perp 4} F_{H\perp 4\perp 4} \wedge \star F_{H\perp 4\perp 4})$.

8.1.1 First term $F_{H44} \wedge \star F_{H44}$

The first term is just $\int F_{H44} \wedge \star F_{H44}$.

Since they are both $SU(2)$ gauge boson terms, this term, in 4-dimensional spacetime, just merges into the $SU(2)$ weak force term $\int F_w \wedge \star F_w$.

8.1.2 Third term $\int_{\perp 4} F_{H\perp 4\perp 4} \wedge \star F_{H\perp 4\perp 4}$

The third term, $\int \int_{\perp 4} F_{H\perp 4\perp 4} \wedge \star F_{H\perp 4\perp 4}$, after integration over $\perp 4$, produces terms of the form

$\lambda(\overline{\Phi}\Phi)^2 - \mu^2\overline{\Phi}\Phi$ by a process similar to the Mayer mechanism (see Mayer's paper [46] for a description of the Mayer mechanism, a geometric Higgs mechanism).

The Mayer mechanism is based on Proposition 11.4 of
chapter 11 of volume I of Kobayashi and Nomizu [42], stating that:
$2F_{H\perp 4\perp 4}(X,Y) = [\Lambda(X), \Lambda(Y)] - \Lambda([X,Y])$,
where Λ takes values in the $SU(2)$ Lie algebra.

If the action of the Hodge dual $\star$ on Λ is such that
$\star\Lambda = -\Lambda$ and $\star[\Lambda,\Lambda] = [\Lambda,\Lambda]$,
then
$F_{H\perp 4\perp 4}(X,Y) \wedge \star F_{H\perp 4\perp 4}(X,Y) = (1/4)([\Lambda(X), \Lambda(Y)]^2 - \Lambda([X,Y])^2)$.

If integration of Λ over $\perp 4$ is $\int_{\perp 4} \Lambda \propto \Phi = (\Phi^+, \Phi^0)$, then

$\int_{\perp 4} F_{H\perp 4\perp 4} \wedge \star F_{H\perp 4\perp 4} = (1/4)\int_{\perp 4}[\Lambda(X), \Lambda(Y)]^2 - \Lambda([X,Y])^2 =$

$= (1/4)[\lambda(\overline{\Phi}\Phi)^2 - \mu^2\overline{\Phi}\Phi]$,

where λ is the strength of the scalar field self-interaction, μ^2 is the other constant in the Higgs potential, and where Φ is a 0-form taking values in the $SU(2)$ Lie algebra.

The $SU(2)$ values of Φ are represented by complex
$SU(2) = Spin(3)$ doublets $\Phi = (\Phi^+, \Phi^0)$.

In real terms, $\Phi^+ = (\Phi_1 + i\Phi_2)/\sqrt{2}$ and $\Phi^0 = (\Phi_3 + i\Phi_4)/\sqrt{2}$,
so Φ has 4 real degrees of freedom.

In terms of real components, $\overline{\Phi}\Phi = (\Phi_1^2 + \Phi_2^2 + \Phi_3^2 + \Phi_4^2)/2$.

The nonzero vacuum expectation value of the
$\lambda(\overline{\Phi}\Phi)^2 - \mu^2\overline{\Phi}\Phi$ term is $v = \mu/\sqrt{\lambda}$, and

$< \Phi^0 > = < \Phi_3 > = v/\sqrt{2}$.

In the unitary gauge, $\Phi_1 = \Phi_2 = \Phi_4 = 0$,
and
$\Phi = (\Phi^+, \Phi^0) = (1/\sqrt{2})(\Phi_1 + i\Phi_2, \Phi_3 + i\Phi_4) = (1/\sqrt{2})(0, v + H)$,

where $\Phi_3 = (v + H)/\sqrt{2}$,
v is the Higgs potential vacuum expectation value, and
H is the real surviving Higgs scalar field.

Since $\lambda = \mu^2/v^2$ and $\Phi = (v + H)/\sqrt{2}$,

$(1/4)[\lambda(\overline{\Phi}\Phi)^2 - \mu^2\overline{\Phi}\Phi] =$

$= (1/16)(\mu^2/v^2)(v + H)^4 - (1/8)\mu^2(v + H)^2 =$

$= (1/16)[\mu^2 v^2 + 4\mu^2 vH + 6\mu^2 H^2 + 4\mu^2 H^3/v + \mu^2 H^4/v^2 - 2\mu^2 v^2 -$
$-4\mu^2 vH - 2\mu^2 H^2] =$

$= (1/4)\mu^2 H^2 - (1/16)\mu^2 v^2 [1 - 4H^3/v^3 - H^4/v^4]$.

8.1.3 Second term $F_{H4\perp4} \wedge \star F_{H4\perp4}$

The second term,
$\int_{\perp4} F_{H4\perp4} \wedge \star F_{H4\perp4}$,
gives $\int \partial\overline{\Phi}\partial\Phi$, by a process similar to the Mayer mechanism (see Mayer's paper [46] for a description of the Mayer mechanism, a geometric Higgs mechanism).

From Proposition 11.4 of chapter 11 of volume I of Kobayashi and Nomizu [42]:
$2F_{H4\perp4}(X, Y) = [\Lambda(X), \Lambda(Y)] - \Lambda([X, Y])$,
where Λ takes values in the $SU(2)$ Lie algebra.

For example, if the X component of $F_{H4\perp4}(X, Y)$ is in the surviving 4 spacetime and the Y component of $F_{H4\perp4}(X, Y)$ is in $\perp$ 4, then the Lie bracket product $[X, Y] = 0$ so that $\Lambda([X, Y]) = 0$ and therefore $F_{H4\perp 4}(X, Y) = (1/2)[\Lambda(X), \Lambda(Y)] = (1/2)\partial_X \Lambda(Y)$.

The total value of $F_{H4\perp 4}(X, Y)$ is then $F_{H4\perp 4}(X, Y) = \partial_X \Lambda(Y)$.

Integration of Λ over $\perp 4$ gives
$\int_{Y\epsilon\perp 4} \partial_X \Lambda(Y) = \partial_X \Phi$,
where, as above, Φ is a 0-form taking values in the $SU(2)$ Lie algebra.

As above, the $SU(2)$ values of Φ are represented by complex $SU(2) = Spin(3)$ doublets $\Phi = (\Phi^+, \Phi^0)$.

In real terms, $\Phi^+ = (\Phi_1 + i\Phi_2)/\sqrt{2}$ and $\Phi^0 = (\Phi_3 + i\Phi_4)/\sqrt{2}$,
so Φ has 4 real degrees of freedom.

As discussed above, in the unitary gauge, $\Phi_1 = \Phi_2 = \Phi_4 = 0$, and
$\Phi = (\Phi^+, \Phi^0) = (1/\sqrt{2})(\Phi_1 + i\Phi_2, \Phi_3 + i\Phi_4) = (1/\sqrt{2})(0, v + H)$,
where $\Phi_3 = (v + H)/\sqrt{2}$,
v is the Higgs potential vacuum expectation value, and
H is the real surviving Higgs scalar field.

The second term is then:
$\int(\int_{\perp 4} -F_{H4\perp 4} \wedge \star F_{H4\perp 4}) =$
$= \int(\int_{\perp 4}(-1/2)[\Lambda(X), \Lambda(Y)] \wedge \star[\Lambda(X), \Lambda(Y)]) = \int \partial\overline{\Phi} \wedge \star\partial\Phi$

where the $SU(2)$ covariant derivative ∂ is
$\partial = \partial + \sqrt{\alpha_w}(W_+ + W_-) + \sqrt{\alpha_w} \cos\theta_w{}^2 W_0$, and θ_w is the Weinberg angle.

Then $\partial\Phi = \partial(v + H)/\sqrt{2} =$
$= [\partial H + \sqrt{\alpha_w} W_+(v + H) + \sqrt{\alpha_w} W_-(v + H) + \sqrt{\alpha_w} W_0(v + H)]/\sqrt{2}$.

In the $D_4 - D_5 - E_6$ model the W_+, W_-, W_0, and H terms are considered to be linearly independent.

$v = v_+ + v_- + v_0$ has linearly independent components v_+, v_-, and v_0 for W_+, W_-, and W_0.

H is the Higgs component.

$\partial\overline{\Phi} \wedge \star\partial\Phi$ is the sum of the squares of the individual terms.

Integration over $\perp 4$ involving two derivatives $\partial_X \partial_X$ is taken to change the sign by $i^2 = -1$.

Then:
$\partial\overline{\Phi} \wedge \star\partial\Phi = (1/2)(\partial H)^2 +$
$+(1/2)[\alpha_w v_+^2 \overline{W_+} W_+ + \alpha_w v_-^2 \overline{W_-} W_- + \alpha_w v_0^2 \overline{W_0} W_0] +$
$+(1/2)[\alpha_w \overline{W_+} W_+ + \alpha_w \overline{W_-} W_- + \alpha_w \overline{W_0} W_0][H^2 + 2vH]$.

Then the full curvature term of the weak-Higgs Lagrangian,
$\int F_w \wedge \star F_w + \partial\overline{\Phi} \wedge \star\partial\Phi + \lambda(\overline{\Phi}\Phi)^2 - \mu^2 \overline{\Phi}\Phi$,

is, by the Higgs mechanism:

$\int [F_w \wedge \star F_w +$
$+(1/2)[\alpha_w v_+^2 \overline{W_+} W_+ + \alpha_w v_-^2 \overline{W_-} W_- + \alpha_w v_0^2 \overline{W_0} W_0] +$
$+(1/2)[\alpha_w \overline{W_+} W_+ + \alpha_w \overline{W_-} W_- + \alpha_w \overline{W_0} W_0][H^2 + 2vH] +$
$+(1/2)(\partial H)^2 + (1/4)\mu^2 H^2 -$
$-(1/16)\mu^2 v^2 [1 - 4H^3/v^3 - H^4/v^4]]$.

The weak boson Higgs mechanism masses, in terms of $v = v_+ + v_- + v_0$, are:

$(\alpha_w/2)v_+^2 = m_{W_+}^2$;

$(\alpha_w/2)v_-^2 = m_{W_-}^2$; and

$(\alpha_w/2)v_0^2 = m_{W_{+0}}^2$,

with $(v = v_+ + v_- + v_0) = ((\sqrt{2})/\sqrt{\alpha_w})(m_{W_+} + m_{W_-} + m_{W_0})$.

Then:

$$\int [F_w \wedge \star F_w +$$
$$+ m_{W_+}^2 W_+ W_+ + m_{W_-}^2 W_- W_- + m_{W_0}^2 W_0 W_0 +$$
$$+(1/2)[\alpha_w \overline{W_+} W_+ + \alpha_w \overline{W_-} W_- + \alpha_w \overline{W_0} W_0][H^2 + 2vH] +$$
$$+(1/2)(\partial H)^2 + (1/2)(\mu^2/2)H^2 -$$
$$-(1/16\mu^2 v^2[1 - 4H^3/v^3 - H^4/v^4].$$

8.2 Gauge Boson Part of the Lagrangian.

In this subsection, we will look at matrix representations of $Spin(0,8)$ and how dimensional reduction affects them.

For a similar study from the point of view of Clifford algebras, see hep-th/9402003 [68]).
For errata for that paper (and others), see
WWW URL http://www.gatech.edu/tsmith/Errata.html [71]).

The gauge boson bivector part of the Lagrangian is

$$\int_{V_8} F_8 \wedge \star F_8 \tag{88}$$

It represents the $D_4 - D_5 - E_6$ model gauge group $Spin(0,8)$ acting in 8-dimensional spacetime.

The 8×8 matrix representation of the $Spin(0,8)$ Lie algebra with the commutator bracket product [,] is

$$\begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \\ -a_{12} & 0 & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} & a_{28} \\ -a_{13} & -a_{23} & 0 & a_{34} & a_{35} & a_{36} & a_{37} & a_{38} \\ -a_{14} & -a_{24} & -a_{34} & 0 & a_{45} & a_{46} & a_{47} & a_{48} \\ -a_{15} & -a_{25} & -a_{35} & -a_{45} & 0 & a_{56} & a_{57} & a_{58} \\ -a_{16} & -a_{26} & -a_{36} & -a_{46} & -a_{56} & 0 & a_{67} & a_{68} \\ -a_{17} & -a_{27} & -a_{37} & -a_{47} & -a_{57} & -a_{67} & 0 & a_{78} \\ -a_{18} & -a_{28} & -a_{38} & -a_{48} & -a_{58} & -a_{68} & -a_{78} & 0 \end{pmatrix} \tag{89}$$

To see how the $Spin(0,8)$ is affected by dimensional reduction of spacetime to 4 dimensions, represent $Spin(0,8)$, as in section 3.2, by

$$\begin{pmatrix} S^7_1 & 0 \\ & \\ 0 & S^7_2 \end{pmatrix} \oplus G_2 \tag{90}$$

This Lie algebra is 7+7+14 = 28-dimensional $Spin(0,8)$, also denoted D_4.

Recall that S^7_1 and S^7_2 represent
the imaginary octonions $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$.
From our present local Lie algebra point of view, they look like linear tangent spaces to 7-spheres, not like global round nonlinear 7-spheres.
Therefore, instead of using the Hopf fibration $S^3 \rightarrow S^7 \rightarrow S^4$,
we break the spaces down in accord with dimensional reduction to:

$$S^7_1 \rightarrow \mathbf{R}^3_1 \oplus \mathbf{R}^4_1 \tag{91}$$

and

$$S^7_2 \rightarrow \mathbf{R}^3_2 \oplus \mathbf{R}^4_2 \tag{92}$$

G_2 has two fibrations:

$$SU(3) \rightarrow G^2 \rightarrow S^6. \tag{93}$$

$$SU(2) \otimes SU(2) \rightarrow G^2 \rightarrow M(G_2)_8 \tag{94}$$

where $M(G_2)_8$ is an 8-dimensional homogeneous rank 2 symmetric space.

By choice of which G_2 fibration to use, $Spin(0, 8)$ has two decompositions from octonionic derivations G_2 to quaternionic derivations $SU(2)$.

First, choose the $SU(3)$ subgroup of G_2 by choosing the G_2 fibration

$$SU(3) \rightarrow G^2 \rightarrow S^6. \tag{95}$$

Since the $SU(3)$ subgroup of G_2 is the larger 8-dimensional part of 14-dimensional G_2, also take the larger $\mathbf{R}^4$ parts of the S^7's, to get:

$$\begin{pmatrix} \mathbf{R}^4_1 & 0 \\ & \\ 0 & \mathbf{R}^4_2 \end{pmatrix} \oplus SU(3) \tag{96}$$

If $\mathbf{R}^4_1 \oplus \mathbf{R}^4_2$ is identified with
the local tangent space of the 8-dimensional manifold

$$(Spin(0,6) \times U(1))/SU(3)$$

then we have constructed
the $Spin(0,6) \times U(1)$ subgroup of $Spin(0,8)$.

The 8-dimensional manifold $(Spin(0,6) \times U(1))/SU(3)$
is built up from the 6-dimensional irreducible symmetric space

$$Spin(0,6)/U(3)$$

(see the book *Einstein Manifolds* by Besse [4])
by adding two $U(1)$'s, one $U(1) = U(3)/SU(3)$ and
the other the $U(1)$ in $Spin(0,6) \times U(1)$.

Now that we have built $Spin(0,6) \times U(1)$ from dimensional reduction process acting on $Spin(0,8)$, compare an 8×8 matrix representation:

$$\begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & 0 & 0 \\ -a_{12} & 0 & a_{23} & a_{24} & a_{25} & a_{26} & 0 & 0 \\ -a_{13} & -a_{23} & 0 & a_{34} & a_{35} & a_{36} & 0 & 0 \\ -a_{14} & -a_{24} & -a_{34} & 0 & a_{45} & a_{46} & 0 & 0 \\ -a_{15} & -a_{25} & -a_{35} & -a_{45} & 0 & a_{56} & 0 & 0 \\ -a_{16} & -a_{26} & -a_{36} & -a_{46} & -a_{56} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{78} \\ 0 & 0 & 0 & 0 & 0 & 0 & -a_{78} & 0 \end{pmatrix} \tag{97}$$

Now make the second choice, the $SU(2) \times SU(2)$ subgroup of G_2 by choosing the G_2 fibration

$$SU(2) \otimes SU(2) \to G^2 \to M(G_2)_8 \tag{98}$$

where $M(G_2)_8$ is an 8-dimensional homogeneous rank 2 symmetric space.

Since the $SU(2) \times SU(2)$ subgroup of G_2 is the smaller 6-dimensional part of 14-dimensional G_2, also take the smaller $\mathbf{R}^3$ parts of the S^7's, to get:

$$\begin{pmatrix} \mathbf{R}^3_1 & 0 \\ & \\ 0 & \mathbf{R}^3_2 \end{pmatrix} \oplus SU(2) \oplus SU(2) \tag{99}$$

If $\mathbf{R}^3_1 \oplus \mathbf{R}^3_2$ is identified with
the local tangent space of the 6-dimensional manifold

$$U(3)/Spin(3)$$

then, since $Spin(3) = SU(2)$, we have constructed
a $U(3) \times SU(2)$ subgroup of $Spin(0,8)$.

The 6-dimensional manifold $U(3)/Spin(3)$
is built up from the 5-dimensional irreducible symmetric space

$$SU(3)/Spin(3)$$

(see the book *Einstein Manifolds* by Besse [4])
by adding the $U(1)$ from $U(1) = U(3)/SU(3)$.

Now that we have built $U(3) \times SU(2)$ from the
dimensional reduction process acting on $Spin(0,8)$,
note that $U(3) = SU(3) \times U(1)$ so that
we have the 12-dimensional Standard Model gauge group

$$SU(3) \times SU(2) \times U(1)$$

Actually, it is even nicer to say that it is

$$U(3) \times SU(2)$$

by putting the electromagnetic $U(1)$ with the color force $SU(3)$, because, as O'Raifeartaigh says in section 9.4 of his book
Group Structure of Gauge Theories [52],
that is the most natural representation of the Standard Model gauge groups. This is because, when fermion representations are taken into account,
the unbroken symmetry is the $U(3)$ of electromagnetism and the color force, while the weak force $SU(2)$ is broken by the Higgs mechanism.

One of the few differences between the Standard Model sector of the 4-dimensional $D_4 - D_5 - E_6$ model and the Standard Model electroweak structure is that
in the $D_4 - D_5 - E_6$ model the electromagnetic $U(1)$ is most naturally put with the color force $SU(3)$, while
in the electroweak Standard Model the electromagnetic $U(1)$ is most naturally put with the weak force $SU(2)$.
As O'Raifearteagh [52] points out, that difference is an advantage of the $D_4 - D_5 - E_6$ model.

Now, look at the Standard Model sector of $D_4 - D_5 - E_6$ model from the matrix representation point of view:

Consider a $U(4)$ subalgebra of $Spin(0, 8)$.

$U(4)$ can be represented (see section 412 G of [22]) as a subalgebra of $Spin(0, 8)$ by

$\mathbf{Re}(U_4)$	$\mathbf{Im}(U_4)$	(100)
$-\mathbf{Im}(U_4)$	$\mathbf{Re}(U_4)$	

Therefore, the $U(4)$ subalgebra of $Spin(0, 8)$ can be represented by

$$\begin{pmatrix} 0 & u_{12} & u_{13} & u_{14} & v_{11} & v_{12} & v_{13} & v_{14} \\ -u_{12} & 0 & u_{23} & u_{24} & v_{12} & v_{22} & v_{23} & v_{24} \\ -u_{13} & -u_{23} & 0 & u_{34} & v_{13} & v_{23} & v_{33} & v_{34} \\ -u_{14} & -u_{24} & -u_{34} & 0 & v_{14} & v_{24} & v_{34} & v_{44} \\ -v_{11} & -v_{12} & -v_{13} & -v_{14} & 0 & u_{12} & u_{13} & u_{14} \\ -v_{12} & -v_{22} & -v_{23} & -v_{24} & -u_{12} & 0 & u_{23} & u_{24} \\ -v_{13} & -v_{23} & -v_{33} & -v_{34} & -u_{13} & -u_{23} & 0 & u_{34} \\ -v_{14} & -v_{24} & -v_{34} & -v_{44} & -u_{14} & -u_{24} & -u_{34} & 0 \end{pmatrix} \tag{101}$$

(Compare this representation of $U(4)$ with the representation above of the isomorphic algebra $Spin(0,6) \times U(1)$.)

A 12-dimensional subalgebra is

$$\begin{pmatrix} 0 & u_{12} & u_{13} & u_{14} & 0 & v_{12} & v_{13} & v_{14} \\ -u_{12} & 0 & u_{23} & u_{24} & v_{12} & 0 & v_{23} & v_{24} \\ -u_{13} & -u_{23} & 0 & u_{34} & v_{13} & v_{23} & 0 & v_{34} \\ -u_{14} & -u_{24} & -u_{34} & 0 & v_{14} & v_{24} & v_{34} & 0 \\ 0 & -v_{12} & -v_{13} & -v_{14} & 0 & u_{12} & u_{13} & u_{14} \\ -v_{12} & 0 & -v_{23} & -v_{24} & -u_{12} & 0 & u_{23} & u_{24} \\ -v_{13} & -v_{23} & 0 & -v_{34} & -u_{13} & -u_{23} & 0 & u_{34} \\ -v_{14} & -v_{24} & -v_{34} & 0 & -u_{14} & -u_{24} & -u_{34} & 0 \end{pmatrix} \tag{102}$$

As we have seen in this subsection, the 28 infinitesimal generators of $Spin(8)$ are broken by dimensional reduction into two parts:

16-dimensional $U(4) = Spin(0,6) \times U(1)$, where $Spin(0,6) = SU(4)$ is the conformal group of 4-dimensional spacetime (the conformal group gives gravity by the MacDowell-Mansouri mechanism [44, 47], and gauge fixing of the conformal group gives the Higgs scalar symmetry breaking) and the $U(1)$ is the complex phase propagators in the 4-dimensional spacetime; and

Physically, the $U(1)$ is the Dirac complexification and it gives the physical Dirac gammas their complex structure, so that they are $\mathbf{C}(4)$ instead of $\mathbf{R}(4)$.

Dirac complexification justifies the physical use of Wick rotation between Euclidean and Minkowski spacetimes, because $\mathbf{C}(4)$ is the Clifford algebra of both the compact Euclidean deSitter Lie group $Spin(0,5)$ and the non-compact Minkowski anti-deSitter Lie group $Spin(2,3)$.

12-dimensional $U(3) \times SU(2)$, where $SU(2)$ is the gauge group of the weak force and $U(3) = SU(3) \times U(1)$ is the $SU(3)$ gauge group of the color force and the $U(1)$ gauge group of electromagnetism.

8.2.1 Conformal Gravity and Higgs Scalar.

$Spin(0,6)$ is the maximal subgroup of $Spin(0,8)$ that acts on the 4-dimensional reduced spacetime.
It acts as the compact version of the conformal group.
An 8×8 matrix representation of $Spin(0,6)$ is

$$\begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & 0 & 0 \\ -a_{12} & 0 & a_{23} & a_{24} & a_{25} & a_{26} & 0 & 0 \\ -a_{13} & -a_{23} & 0 & a_{34} & a_{35} & a_{36} & 0 & 0 \\ -a_{14} & -a_{24} & -a_{34} & 0 & a_{45} & a_{46} & 0 & 0 \\ -a_{15} & -a_{25} & -a_{35} & -a_{45} & 0 & a_{56} & 0 & 0 \\ -a_{16} & -a_{26} & -a_{36} & -a_{46} & -a_{56} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{103}$$

The MacDowell-Mansouri mechanism [44] produces a classical model of gravity from a $Spin(0,5)$ de Sitter gauge group. As a subalgebra of the $Spin(0,6)$ Lie algebra, the $Spin(0,5)$ Lie algebra can be represented by

$$\begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} & a_{15} & 0 & 0 & 0 \\ -a_{12} & 0 & a_{23} & a_{24} & a_{25} & 0 & 0 & 0 \\ -a_{13} & -a_{23} & 0 & a_{34} & a_{35} & 0 & 0 & 0 \\ -a_{14} & -a_{24} & -a_{34} & 0 & a_{45} & 0 & 0 & 0 \\ -a_{15} & -a_{25} & -a_{35} & -a_{45} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{104}$$

The 5 elements $\{a_{16}, a_{26}, a_{36}, a_{46}, a_{56}\}$ that are in $Spin(0,6)$ but not in $Spin(0,5)$ are the 1 scale and 4 conformal degrees of freedom that are gauge-fixed by Mohapatra in section 14.6 of [47] to get from the 15-dimensional conformal group $Spin(0,6)$ to the 10-dimensional deSitter group $Spin(0,5)$ so that the MacDowell-Mansouri mechanism [44] can be used to produce Einstein-Hilbert gravity plus a cosmological constant term, an Euler topological term, and a Pontrjagin topological term.
As Nieto, Obregon, and Socorro [51] have shown, MacDowell-Mansouri de-

Sitter gravity is equivalent to Ashtekar gravity plus a cosmological constant term, an Euler topological term, and a Pontrjagin topological term.

For further discussion of the MacDowell-Mansouri mechanism,
see Freund [27] (chapter 21), or Ne'eman and Regge [50](at pages 25-28),
or Nieto, Obregon, and Socorro [51]

The physical reason for fixing the scale and conformal degrees of freedom lies in the relationship between gravity and the Higgs mechanism.

Since all rest mass comes from the Higgs mechanism, and since rest mass interacts through gravity, it is natural for gravity and Higgs symmetry breaking to be related at a fundamental level.

As remarked by Sardanashvily [61], Heisenberg and Ivanenko in the 1960s made the first atttempt to connect gravity with a symmetry breaking mechanism by proposing that the graviton might be a Goldstone boson resulting from breaking Lorentz symmetry in going from flat MInkowski spacetime to curved spacetime.

Sardanashvily [61] (see also gr-qc/9405013 [59] gr-qc/9407032 [60]
gr-qc/9411013 [62]) proposes that gravity be represented by a gauge theory with group $GL(4)$, that $GL(4)$ symmetry can be broken to either Lorentz $SO(3,1)$ symmetry or $SO(4)$ symmetry, and that the resulting Higgs fields can be interpreted as either the gravitational field (for breaking to $SO(3,1)$ or the Riemannian metric (for breaking to $SO(4)$.

The identification of a pseudo-Riemannian metric with a Higgs field was made by Trautman [74], by Sardanashvily [58] , and by Ivanenko and Sardanashvily [41]

In the $D_4-D_5-E_6$ model (using here the compact version) the conformal group $Spin(0,6) = SU(4)$ is broken to the de Sitter group $Spin(0,5) = Sp(2)$ by fixing the 1 scale and 4 conformal gauge degrees of freedom.

The resulting Higgs field is interpreted in the $D_4 - D_5 - E_6$ model as the same Higgs field that gives mass to the $SU(2)$ weak bosons and to the Dirac fermions by the Higgs mechanism.

The Higgs mechanism requires "spontaneous symmetry breaking" of a scalar field potential whose minima are not zero, but which form a 3-sphere $S^3 = SU(2)$.

In particular, one real component of the complex Higgs scalar doublet is set to $v/\sqrt{2})$, where v is the modulus of the S^3 of minima, usually called the vacuum expectation value.

If the S^3 is taken to be the unit quaternions, then the "spontaneous symmetry breaking" requires choosing a (positive) real axis for the quaternion space.

In the standard model, it is assumed that a random vacuum fluctuation breaks the $SU(2)$ symmetry and in effect chooses a real axis at random.

In the $D_4 - D_5 - E_6$ model, the symmetry breaking from conformal $Spin(0,6)$ to de Sitter $Spin(0,5)$ by fixing the 1 scale and 4 conformal gauge degrees of freedom is a symmetry breaking mechanism that does not require perturbation by a random vacuum fluctuation.

Gauge-fixing the 1 scale degree of freedom fixes a length scale. It can be chosen to be the magnitude of the vacuum expectation value, or radius of the S^3.

Gauge-fixing the 4 conformal degrees of freedom fixes the (positive) real axis of the S^3 consistently throughout 4-dimensional spacetime.

Therefore, the $D_4 - D_5 - E_6$ model Higgs field comes from the breaking of $Spin(0,6)$ conformal symmetry to $Spin(0,5)$ de Sitter gauge symmetry, from which Einstein-Hilbert gravity can be constructed by the MacDowell-Mansouri mechanism.

Einstein-Hilbert gravity as a spin-2 field theory in flat spacetime: Feynman, in his 1962-63 lectures at Caltech [24], showed how Einstein-Hilbert gravity can be described by starting with a linear spin-2 field theory in flat spacetime, and then adding higher-order terms to get Einstein-Hilbert gravity. The observed curved spacetime is based on an unobservable flat spacetime. (see also Deser [14]

The Feynman spin-2 flat spacetime construction of Einstein-Hilbert gravity allows the $D_4 - D_5 - E_6$ model to be based on a fundamental D_4 lattice 4-dimensional spacetime.

Quantization in the $D_4 - D_5 - E_6$ model is fundamentally based on a path integral sum over histories of paths in a D_4 lattice spacetime using a generalized Feynman checkerboard.

The generalized Feynman checkerboard is discussed at

WWW URL http://www.gatech.edu/tsmith/FynCkbd.html [71].

Fundamentally, that is nice, but calculations can be very difficult, particularly for quantum gravity.

The work of Garcia-Compean et. al [29] suggests that the most practical approach to quantum gravity may be through BRST symmetry.

Quantization breaks the gauge group invariance of the $D_4 - D_5 - E_6$ model Lagrangian, because the path integral must not overcount paths by including more than one representative of each gauge-equivalence class of paths.
The remaining quantum symmetry is the symmetry of BRST cohomology classes.
Knowledge of the BRST symmetry tells you which ghosts must be used in quantum calculations, so the BRST cohomology can be taken to be the basis for the quantum theory.

Garcia-Compean et. al [29] discuss two current approaches to quantum gravity:

string theory, which abandons point particles even at the classical level; and

redefinition of classical general relativity in terms of new variables, the Ashtekar variables [3] and trying to use the new variables to construct a quantum theory of gravity.

Nieto, Obregon, and Socorro [51] have shown
that the MacDowell-Mansouri $Spin(0,5) = Sp(2)$ de Sitter Lagrangian for
gravity used in the $D_4 - D_5 - E_6$ model is equal to
the Lagrangian for gravity in terms of the
Ashtekar variables [3] plus

a cosmological constant term,

an Euler topological term, and

a Pontrjagin topological term.

Therefore, although the quantum gravity methods of string theory cannot be used in the $D_4 - D_5 - E_6$ model because the $D_4 - D_5 - E_6$ model uses fundamental point particles at the classical level,

the methods based on Ashtekar variables [3] are available.

Two such approaches are:

a topological approach based on loop groups; and

an algebraic approach based on getting BRST transformations
from Maurer-Cartan horizontality conditions.

The latter approach, which is described in Blaga, et. al. [5]
is the approach used for the $D_4 - D_5 - E_6$ model.

8.2.2 Chern-Simons Time

An essential part of a quantum theory of gravity is the correct definition of physical time.

Nieto, Obregon, and Socorro [51] have shown that Lagrangian action of the Ashtekar variables [3] is a Chern-Simons action if the Killing metric of the de Sitter group is used instead of the Levi-Civita tensor.

Smolin and Soo [72] have shown that the Chern-Simons invariant of the Ashtekar-Sen connection is a natural candidate for the internal time coordinate for classical and quantum cosmology, so that the $D_4 - D_5 - E_6$ model uses Chern-Simons time.

8.2.3 Quantum Gravity plus Standard Model

Another essential part of a quantum theory of gravity is the correct relationship of quantum gravity with the quantum theory of the forces and particles of the Standard Model, to calculate how standard model particles and fields interact in the presence of gravity.

Moritsch, et. al. [48] have done this by using Maurer-Cartan horizontality conditions to get BRST transformations for Yang-Mills gauge fields in the presence of gravity.

8.2.4 Color, Weak, and Electromagnetic Forces.

12-dimensional $U(3) \times SU(2)$, where $SU(2)$ is the gauge group of the weak force and $U(3) = SU(3) \times U(1)$ is the $SU(3)$ gauge group of the color force and the $U(1)$ gauge group of electromagnetism, can be represented in terms of 8×8 matrices by

$$\begin{pmatrix} 0 & u_{12} & u_{13} & u_{14} & 0 & v_{12} & v_{13} & v_{14} \\ -u_{12} & 0 & u_{23} & u_{24} & v_{12} & 0 & v_{23} & v_{24} \\ -u_{13} & -u_{23} & 0 & u_{34} & v_{13} & v_{23} & 0 & v_{34} \\ -u_{14} & -u_{24} & -u_{34} & 0 & v_{14} & v_{24} & v_{34} & 0 \\ 0 & -v_{12} & -v_{13} & -v_{14} & 0 & u_{12} & u_{13} & u_{14} \\ -v_{12} & 0 & -v_{23} & -v_{24} & -u_{12} & 0 & u_{23} & u_{24} \\ -v_{13} & -v_{23} & 0 & -v_{34} & -u_{13} & -u_{23} & 0 & u_{34} \\ -v_{14} & -v_{24} & -v_{34} & 0 & -u_{14} & -u_{24} & -u_{34} & 0 \end{pmatrix} \tag{105}$$

In terms of the fibrations of S^7 and G_2, the 12-dimensional subalgebra $U(3) \times SU(2)$ is represented by

$$\begin{pmatrix} S^3_a & 0 \\ 0 & S^3_b \end{pmatrix} \oplus SU(2) \oplus SU(2) \tag{106}$$

The result of this decomposition is that the gauge boson bivector part of the 8-dimensional $D_4 - D_5 - E_6$ Lagrangian

$$\int_{V_8} F_{Spin(0,8)} \wedge \star F_{Spin(0,8)} \tag{107}$$

breaks down into

$$\int_{V_4} F_{Spin(0,6)} \wedge \star F_{Spin(0,6)} \oplus F_{U(1)} \wedge \star F_{U(1)} \oplus F_{SU(3)} \wedge \star F_{SU(3)} \oplus F_{SU(2)} \wedge \star F_{SU(2)} \tag{108}$$

The 28 gauge bosons of 8-dimensional $Spin(0,8)$ are broken into four independent (or commuting, from the Lie algebra point of view) sets of gauge bosons:

15 for gravity and Higgs symmetry breaking $Spin(0,6)$, plus 1 for the $U(1)$ propagator phase;

1 for $U(1)$ electromagentism;

8 for color $SU(3)$; and

3 for the $SU(2)$ weak force.

Each of the terms of the form

$$\int_{V_4} F \wedge \star F \tag{109}$$

contains a force strength constant.

The force strength constants define the relative strengths of the four forces.

One of the factors determining the force strength constants is the relative magnitude of the measures of integration over the 4-dimensional spacetime base manifold in each integral.

The relative magnitude of the measures is proportional to the volume $Vol(M_{force})$ of the irreducible m_{force}(real)-dimensional symmetric space on which the gauge group acts naturally as a component of 4(real) dimensional spacetime $M_{force}^{\left(\frac{4}{m_{force}}\right)}$.

The M_{force} manifolds for the gauge groups of the four forces are:

$Gauge\,Group$	$Symmetric\,Space$	m_{force}	M_{force}
$Spin(5)$	$\frac{Spin(5)}{Spin(4)}$	4	S^4
$SU(3)$	$\frac{SU(3)}{SU(2)\times U(1)}$	4	$\mathbf{C}P^2$
$SU(2)$	$\frac{SU(2)}{U(1)}$	2	$S^2 \times S^2$
$U(1)$	$U(1)$	1	$S^1 \times S^1 \times S^1 \times S^1$

(110)

Further discussion of this factor is in

WWW URL http://www.gatech.edu/tsmith/See.html [71].

The second factor in the force strengths is based on the interaction of the gauge bosons with the spinor fermions through the covariant derivative.

When the spinor fermion term is added to the 4-dimensional $D_4 - D_5 - E_6$ Lagrangian for each force, you get a Lagrangian of the form

$$\int_{V_4} F \wedge \star F + \overline{S_{8\pm}}\ \partial\!\!\!/_8 S_{8\pm} \tag{111}$$

The covariant derivative part of the Dirac operator $\partial\!\!\!/_8$ gives the interaction between each of the four forces and the spinor fermions.

The strength of each force depends on the magnitude of the interaction of the covariant derivative of the force with the spinor fermions. Since the spinor fermions are defined with respect to a space $Q = S^7 \times \mathbf{R}P^1$ that is the Shilov boundary of a bounded complex homogeneous domain D, the relative strength of each force can be measured by the relative volumes of the part of the manifolds Q and D that are affected by that force.

Let $Vol(Q_{force})$ be the volume of that part of the full compact fermion state space manifold $\mathbf{R}P^1 \times S^7$ on which a gauge group acts naturally through its charged (color or electromagnetic charge) gauge bosons.

For the forces with charged gauge bosons,

$Spin(5)$ gravity,

$SU(3)$ color force, and

$SU(2)$ weak force,

Q_{force} is the Shilov boundary of the bounded complex homogeneous domain D_{force} that corresponds to the Hermitian symmetric space on which the gauge group acts naturally as a local isotropy (gauge) group.

For $U(1)$ electromagnetism, whose photon carries no charge, the factors $Vol(Q_{U(1)})$ and $Vol(D_{U(1)})$ do not apply and are set equal to 1.

The volumes $Vol(M_{force})$, $Vol(Q_{force})$, and $Vol(D_{force})$ are calculated with $M_{force}, Q_{force}, D_{force}$ normalized to unit radius.

The factor $\frac{1}{Vol(D_{force})^{\left(\frac{1}{m_{force}}\right)}}$ is a normalization factor to be used if the dimension of Q_{force} is different from the dimension m_{force}, in order to normalize the radius of Q_{force} to be consistent with the unit radius of M_{force}.

The Q_{force} and D_{force} manifolds for the gauge groups of the four forces are:

Gauge Group	*Hermitian Symmetric Space*	*Type of* D_{force}	m_{force}	Q_{force}
$Spin(5)$	$\frac{Spin(7)}{Spin(5)\times U(1)}$	IV_5	4	$\mathbf{R}P^1 \times S^4$
$SU(3)$	$\frac{SU(4)}{SU(3)\times U(1)}$	$B^6\ (ball)$	4	S^5
$SU(2)$	$\frac{Spin(5)}{SU(2)\times U(1)}$	IV_3	2	$\mathbf{R}P^1 \times S^2$
$U(1)$	$-$	$-$	1	$-$

(112)

The third factor affects only the force of gravity, which has a characteristic mass because the Planck length is the fundamental lattice length in the $D_4 - D_5 - E_6$ model, so that $\mu_{Spin(0,5)} = M_{Planck}$ and
the weak force, whose gauge bosons acquire mass by the Higgs mechanism, so that $\mu_{Spin(0,5)} = \sqrt{m_{W+}^2 + m_{W-}^2 + m_{W_0}^2}$.

For the weak force, the relevant factor is

$$\frac{1}{\mu_{force}^2} = \frac{1}{m_{W+}^2 + m_{W-}^2 + m_{W_0}^2}$$

For gravity, it is

$$\frac{1}{\mu_{force}^2} = \frac{1}{m_{Planck}^2}$$

For the $SU(3)$ color and $U(1)$ electromagnetic forces,

$$\frac{1}{\mu_{force}^2} = 1$$

Taking all the factors into account, the calculated strength of a force is taken to be proportional to the product:

$$\left(\frac{1}{\mu_{force}^2}\right)(Vol(M_{force}))\left(\frac{Vol(Q_{force})}{Vol(D_{force})^{\left(\frac{1}{m_{force}}\right)}}\right) \tag{113}$$

The geometric force strengths, that is,
everything but the mass scale factors $1/\mu_{force}^2$,
normalized by dividing them by the largest one,
the one for gravity.

The geometric volumes needed for the calculations, mostly taken from Hua [40], are

$Force$	M	$Vol(M)$	Q	$Vol(Q)$	D	$Vol(D$
$gravity$	S^4	$8\pi^2/3$	$\mathbf{R}P^1 \times S^4$	$8\pi^3/3$	IV_5	$\pi^5/2^4 5!$
$color$	$\mathbf{C}P^2$	$8\pi^2/3$	S^5	$4\pi^3$	B^6 $(ball)$	$\pi^3/6$
$weak$	$S^2 \times S^2$	$2 \times 4\pi$	$\mathbf{R}P^1 \times S^2$	$4\pi^2$	IV_3	$\pi^3/24$
$e-mag$	T^4	$4 \times 2\pi$	$-$	$-$	$-$	$-$

(114)

Using these numbers, the results of the calculations are the relative force strengths at the characteristic energy level of the generalized Bohr radius of each force:

Gauge Group	*Force*	*Characteristic Energy*	*Geometric Force Strength*	*Total Force Strength*
$Spin(5)$	*gravity*	$\approx 10^{19} GeV$	1	$G_G m_{proton}^2 \approx 5 \times 10^{-39}$
$SU(3)$	*color*	$\approx 245 MeV$	0.6286	0.6286
$SU(2)$	*weak*	$\approx 100 GeV$	0.2535	$G_W m_{proton}^2 \approx \approx 1.02 \times 10^{-5}$
$U(1)$	$e - mag$	$\approx 4 KeV$	1/137.03608	1/137.03608

(115)

The force strengths are given at the characteristic energy levels of their forces, because the force strengths run with changing energy levels.

The effect is particularly pronounced with the color force.

In WWW URL http://www.gatech.edu/tsmith/cweRen.html [71],
the color force strength was calculated at various energies according to renormalization group equations, with the following results:

Energy Level	*Color Force Strength*
$245 MeV$	0.6286
$5.3 GeV$	0.166
$34 GeV$	0.121
$91 GeV$	0.106

(116)

Shifman WWW URL http://xxx.lan.gov/abs/hep-ph/9501222 [65] has noted that Standard Model global fits at the Z peak, about 91 GeV, give a color

force strength of about 0.125 with $\Lambda_{QCD} \approx 500\ MeV$,
whereas low energy results and lattice calculations give a color force strength at the Z peak of about 0.11 with $\Lambda_{QCD} \approx 200\ MeV$.

The low energy results and lattice calculations are closer to the tree level $D_4 - D_5 - E_6$ model value at 91 GeV of 0.106.
Also, the $D_4 - D_5 - E_6$ model has $\Lambda_{QCD} \approx 245\ MeV$
(For the pion mass, upon which the Λ_{QCD} calculation depends, see WWW URL http://www.gatech.edu/tsmith/SnGdnPion.html [71].)

8.3 Fermion Part of the Lagrangian

Consider the spinor fermion term $\int \overline{S_{8\pm}} \not\partial_8 S_{8\pm}$

For each of the surviving 4-dimensional 4 and reduced 4-dimensional $\perp 4$ of 8-dimensional spacetime, the part of $S_{8\pm}$ on which the Higgs $SU(2)$ acts locally is $Q_3 = \mathbf{R}P^1 \times S^2$.

It is the Shilov boundary of the bounded domain D_3 that is isomorphic to the symmetric space $\overline{D_3} = Spin(5)/SU(2) \times U(1)$.

The Dirac operator $\not\partial_8$ decomposes as $\not\partial = \not\partial_4 + \not\partial_{\perp 4}$, where
$\not\partial_4$ is the Dirac operator corresponding to the surviving spacetime 4 and
$\not\partial_{\perp 4}$ is the Dirac operator corresponding to the reduced $4 \perp 4$.

Then the spinor term is $\int \overline{S_{8\pm}} \not\partial_4 S_{8\pm} + \overline{S_{8\pm}} \not\partial_{\perp 4} S_{8\pm}$
The Dirac operator term $\not\partial_{\perp 4}$ in the reduced $\perp 4$ has dimension of mass.

After integration $\int \overline{S_{8\pm}} \not\partial_{\perp 4} S_{8\pm}$ over the reduced $\perp 4$,
$\not\partial_{\perp 4}$ becomes the real scalar Higgs scalar field $Y = (v + H)$ that comes from the complex $SU(2)$ doublet Φ after action of the Higgs mechanism.

If integration over the reduced $\perp 4$ involving two fermion terms $\overline{S_{8\pm}}$ and $S_{8\pm}$ is taken to change the sign by $i^2 = -1$, then, by the Higgs mechanism,
$\int \overline{S_{8\pm}} \not\partial_{\perp 4} S_{8\pm} \rightarrow \int (\int_{\perp 4} \overline{S_{8\pm}} \not\partial_{\perp 4} S_{8\pm}) \rightarrow$
$\rightarrow - \int \overline{S_{8\pm}} Y Y S_{8\pm} = - \int \overline{S_{8\pm}} Y (v + H) S_{8\pm}$,

where:

H is the real physical Higgs scalar, $m_H = v\sqrt{(}\lambda/2)$, and v is the vacuum expectation value of the scalar field Y, the free parameter in the theory that sets the mass scale.

Denote the sum of the three weak boson masses by Σ_{m_W}.

$v = \Sigma_{m_W}((\sqrt{2})/\sqrt{\alpha_w}) = 260.774 \times \sqrt{2}/0.5034458 = 732.53\ GeV$,
a value chosen so that the electron mass will be 0.5110 MeV.

The Higgs vacuum expectation value $v = (v_+ + v_- + v_0)$ is the only particle mass free parameter.

In the $D_4 - D_5 - E_6$ model, v is set so that the electron mass $m_e = 0.5110 MeV$.

In the $D_4 - D_5 - E_6$ model, α_w is calculated to be $\alpha_w = 0.2534577$, so $\sqrt{\alpha_w} = 0.5034458$ and $v = 732.53$ GeV.

The Higgs mass m_H is given by the term

$$(1/2)(\partial H)^2 - (1/2)(\mu^2/2)H^2 = (1/2)[(\partial H)^2 - (\mu^2/2)H^2] \qquad (117)$$

to be

$$m_H^2 = \mu^2/2 = \lambda v^2/2 \qquad (118)$$

so that

$$m_H = \sqrt{(\mu^2/2)} = \sqrt{\lambda}v^2/2) \qquad (119)$$

λ is the scalar self-interaction strength.
It should be the product of the "weak charges" of two scalars coming from the reduced 4 dimensions in $Spin(4)$, which should be the same as the weak charge of the surviving weak force $SU(2)$ and therefore just the square of the $SU(2)$ weak charge, $\sqrt{(\alpha_w^2)} = \alpha_w$, where α_w is the $SU(2)$ geometric force strength.

Therefore $\lambda = \alpha_w = 0.2534576$, $\sqrt{\lambda} = 0.5034458$, and $v = 732.53$ GeV, so that the mass of the Higgs scalar is

$$m_H = v\sqrt{(\lambda/2)} = 260.774\ GeV. \qquad (120)$$

Y is the Yukawa coupling between fermions and the Higgs field.

Y acts on all 28 elements (2 helicity states for each of the 7 Dirac particles and 7 Dirac antiparticles) of the Dirac fermions in a given generation, because all of them are in the same $Spin(0, 8)$ spinor representation.

8.3.1 Calculation of Particle Masses.

Denote the sum of the first generation Dirac fermion masses by Σ_{f_1}.

Then $Y = (\sqrt{2})\Sigma_{f_1}/v$, just as $\sqrt{(\alpha_w)} = (\sqrt{2})\Sigma_{m_W}/v$.

Y should be the product of two factors:

e^2, the square of the electromagnetic charge $e = \sqrt{\alpha_E}$, because in the term $\int(\int_{\perp 4} \overline{S_{8\pm}} \ \not{\partial}_{\perp 4} S_{8\pm}) \rightarrow -\int \overline{S_{8\pm}} Y(v+H) S_{8\pm}$ each of the Dirac fermions $S_{8\pm}$ carries electromagnetic charge proportional to e ; and

$1/g_w$, the reciprocal of the weak charge $g_w = \sqrt{\alpha_w}$, because an $SU(2)$ force, the Higgs $SU(2)$, couples the scalar field to the fermions.

Therefore

$$\Sigma_{f_1} = Yv/\sqrt{2} = (e^2/g_w)v/\sqrt{2} = 7.508 \ GeV \tag{121}$$

and

$$\Sigma_{f_1}/\Sigma_{m_W} = (e^2/g_w)v/g_w v = e^2/g_w^2 = \alpha_E/\alpha_w \tag{122}$$

The Higgs term $-\int \overline{S_{8\pm}} Y(v+H) \ S_{8\pm} = -\int \overline{S_{8\pm}} \ Yv S_{8\pm} - \int \overline{S_{8\pm}} YHS_{8\pm} =$ $= -\int \overline{S_{8\pm}}(\sqrt{2}\Sigma_{f_1})S_{8\pm} - \int \overline{S_{8\pm}}(\sqrt{2}\Sigma_{f_1}/v)S_{8\pm}$.

The resulting spinor term is of the form $\int[\overline{S_{8\pm}}(\not{\partial} - Yv)S_{8\pm} - \overline{S_{8\pm}} YHS_{8\pm}]$ where $(\not{\partial} - Yv)$ is a massive Dirac operator.

How much of the total mass $\Sigma_{f_1} = Yv/\sqrt{2} = 7.5 \ GeV$ is allocated to each of the first generation Dirac fermions is determined by calculating the individual fermion masses in the $D_4 - D_5 - E_6$ model, and

those calculations also give the values of

$$\Sigma_{f_2} = 32.9 \ GeV \tag{123}$$

$$\Sigma_{f_3} = 1,629 \ GeV \tag{124}$$

as well as second and third generation individual fermion masses, with the result that the individual tree-level lepton masses and quark constituent masses are:

$m_e = 0.5110$ MeV (assumed);
$m_{\nu_e} = m_{\nu_\mu} = m_{\nu_\tau} = 0$;
$m_d = m_u = 312.8$ MeV (constituent quark mass);
$m_\mu = 104.8$ MeV;
$m_s = 625$ MeV (constituent quark mass);
$m_c = 2.09$ GeV (constituent quark mass);
$m_\tau = 1.88$ GeV;
$m_b = 5.63$ GeV (constituent quark mass); v

$$m_t \ = \ 130 \ GeV \ (constituent \ quark \ mass). \tag{125}$$

Here is how the individual fermion mass calculations are done in the $D_4 - D_5 - E_6$ model.

The Weyl fermion neutrino has at tree level only the left-handed state, whereas the Dirac fermion electron and quarks can have both left-handed and right-handed states, so that the total number of states corresponding to each of the half-spinor $Spin(0, 8)$ representations is 15.

Neutrinos are massless at tree level in all generations.

In the $D_4 - D_5 - E_6$model, the first generation fermions correspond to octonions $\mathbf{O}$, while second generation fermions correspond to pairs of octonions $\mathbf{O} \times \mathbf{O}$ and third generation fermions correspond to triples of octonions $\mathbf{O} \times \mathbf{O} \times \mathbf{O}$.

To calculate the fermion masses in the model, the volume of a compact manifold representing the spinor fermions S_{8+} is used. It is the parallelizable manifold $S^7 \times RP^1$.

Also, since gravitation is coupled to mass, the infinitesimal generators of the MacDowell-Mansouri gravitation group, $Spin(0, 5)$, are relevant.

The calculated quark masses are constituent masses, not current masses.

In the $D_4 - D_5 - E_6$ model, fermion masses are calculated as a product of four factors:

$$V(Q_{fermion}) \times N(Graviton) \times N(octonion) \times Sym \tag{126}$$

$V(Q_{fermion})$ is the volume of the part of the half-spinor
fermion particle manifold $S^7 \times RP^1$ that is
related to the fermion particle by photon, weak boson, and gluon interactions.

$N(Graviton)$ is the number of types of $Spin(0,5)$ graviton related to the fermion. The 10 gravitons correspond to the 10 infinitesimal generators of $Spin(0,5) = Sp(2)$.
2 of them are in the Cartan subalgebra.
6 of them carry color charge, and may therefore be considered as corresponding to quarks.
The remaining 2 carry no color charge, but may carry electric charge and so may be considered as corresponding to electrons.
One graviton takes the electron into itself, and the other can only take the first-generation electron into the massless electron neutrino.
Therefore only one graviton should correspond to the mass of the first-generation electron.
The graviton number ratio of the down quark to the first-generation electron is therefore $6/1 = 6$.

$N(octonion)$ is an octonion number factor relating up-type quark masses to down-type quark masses in each generation.

Sym is an internal symmetry factor, relating 2nd and 3rd generation massive leptons to first generation fermions.
It is not used in first-generation calculations.

The ratio of the down quark constituent mass to the electron mass is then calculated as follows:
Consider the electron, e.
By photon, weak boson, and gluon interactions, e can only be taken into 1, the massless neutrino.
The electron and neutrino, or their antiparticles, cannot be combined to produce any of the massive up or down quarks.

The neutrino, being massless at tree level, does not add anything to the mass formula for the electron.
Since the electron cannot be related to any other massive Dirac fermion, its volume $V(Q_{electron})$ is taken to be 1.

Next consider a red down quark e_3.
By gluon interactions, e_3 can be taken into e_5 and e_7, the blue and green down quarks.
By weak boson interactions, it can be taken into e_1, e_2, and e_6, the red, blue, and green up quarks.
Given the up and down quarks, pions can be formed from quark-antiquark pairs, and the pions can decay to produce electrons and neutrinos.
Therefore the red down quark (similarly, any down quark) is related to any part of $S^7 \times \mathbf{R}P^1$, the compact manifold corresponding to

$$\{1, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$$

and therefore a down quark should have a spinor manifold volume factor $V(Q_{downquark}$ of the volume of $S^7 \times \mathbf{R}P^1$.
The ratio of the down quark spinor manifold volume factor to the electron spinor manifold volume factor is just

$$V(Q_{downquark})/V(Q_{electron}) = V(S^7 \times \mathbf{R}P^1)/1 = \pi^5/3. \qquad (127)$$

Since the first generation graviton factor is 6,

$$md/me = 6V(S^7 \times \mathbf{R}P^1) = 2\pi^5 = 612.03937 \qquad (128)$$

As the up quarks correspond to e_1, e_2, and e_6, which are isomorphic to e_3, e_5, and e_7 of the down quarks, the up quarks and down quarks have the same constituent mass $m_u = m_d$.

Antiparticles have the same mass as the corresponding particles.

Since the model only gives ratios of massses, the mass scale is fixed by assuming that the electron mass $m_e = 0.5110$ MeV.

Then, the constituent mass of the down quark is $m_d = 312.75$ MeV, and the constituent mass for the up quark is $m_u = 312.75$ MeV.

As the proton mass is taken to be the sum of the constituent masses of its constituent quarks

$$m_{proton} = m_u + m_u + m_d = 938.25\ MeV \tag{129}$$

The $D_4 - D_5 - E_6$ model calculation is close to the experimental value of 938.27 MeV.

The third generation fermion particles correspond to triples of octonions. There are $8^3 = 512$ such triples.

The triple $\{1, 1, 1\}$ corresponds to the tau-neutrino.

The other 7 triples involving only 1 and e_4 correspond to the tauon:

$$\{e_4, e_4, e_4\}, \{e_4, e_4, 1\}, \{e_4, 1, e_4\}, \{1, e_4, e_4\}, \{1, 1, e_4\}, \{1, e_4, 1\}, \{e_4, 1, 1\}$$

,

The symmetry of the 7 tauon triples is the same as the symmetry of the 3 down quarks, the 3 up quarks, and the electron, so the tauon mass should be the same as the sum of the masses of the first generation massive fermion particles.

Therefore the tauon mass 1.87704 GeV.

Note that all triples corresponding to the tau and the tau-neutrino are colorless.

The beauty quark corresponds to 21 triples.
They are triples of the same form as the 7 tauon triples, but for 1 and e_3, 1 and e_5, and 1 ande_7, which correspond to the red, green, and blue beauty quarks, respectively.

The seven triples of the red beauty quark correspond to the seven triples of the tauon, except that the beauty quark interacts with 6 $Spin(0, 5)$ gravitons while the tauon interacts with only two.

The beauty quark constituent mass should be the tauon mass times the third generation graviton factor $6/2 = 3$, so the B-quark mass is
$m_b = 5.63111$ GeV.

Note particularly that triples of the type $\{1, e_3, e_5\}$, $\{e_3, e_5, e_7\}$, etc., do not correspond to the beauty quark, but to the truth quark.

The truth quark corresponds to the remaining 483 triples, so the constituent mass of the red truth quark is $161/7 = 23$ times the red beauty quark mass, and the red T-quark mass is

$$m_t = 129.5155\ GeV \tag{130}$$

The blue and green truth quarks are defined similarly.

The tree level T-quark constituent mass rounds off to 130 GeV.

These results when added up give a total mass of
third generation fermions:

$$\Sigma_{f_3} = 1,629\ GeV \tag{131}$$

The second generation fermion calculations are:

The second generation fermion particles correspond to pairs of octonions. There are 82 = 64 such pairs.
The pair $\{1, 1\}$ corresponds to the μ-neutrino.
The pairs $\{1, e_4\}$, $\{e_4, 1\}$, and $\{e_4, e_4\}$ correspond to the muon.
Compare the symmetries of the muon pairs to the symmetries of the first generation fermion particles.
The pair $\{e_4, e_4\}$ should correspond to the e_4 electron.
The other two muon pairs have a symmetry group S2, which is 1/3 the size of the color symmetry group S3 which gives the up and down quarks their mass of 312.75 MeV.

Therefore the mass of the muon should be the sum of the $\{e_4, e_4\}$ electron mass and the $\{1, e_4\}$, $\{e_4, 1\}$ symmetry mass, which is 1/3 of the up or down quark mass.

Therefore, $m_\mu = 104.76$ MeV.

Note that all pairs corresponding to the muon and the μ-neutrino are colorless.

The red, blue and green strange quark each corresponds to the 3 pairs involving 1 and e_3, e_5, or e_7.

The red strange quark is defined as the three pairs 1 and e_3, because e_3 is the red down quark.
Its mass should be the sum of two parts: the $\{e_3, e_3\}$ red down quark mass, 312.75 MeV, and the product of the symmetry part of the muon mass, 104.25 MeV, times the graviton factor.

Unlike the first generation situation, massive second and third generation leptons can be taken, by both of the colorless gravitons that may carry electric charge, into massive particles.

Therefore the graviton factor for the second and third generations is $6/2 = 3$.

Therefore the symmetry part of the muon mass times the graviton factor 3 is 312.75 MeV, and the red strange quark constituent mass is

$$m_s = 312.75\ MeV + 312.75\ MeV = 625.5\ MeV$$

The blue strange quarks correspond to the three pairs involving e_5, the green strange quarks correspond to the three pairs involving e_7, and their masses are determined similarly.

The charm quark corresponds to the other 51 pairs. Therefore, the mass of the red charm quark should be the sum of two parts:

the $\{e_1, e_1\}$, red up quark mass, 312.75 MeV; and

the product of the symmetry part of the strange quark mass, 312.75 MeV, and

the charm to strange octonion number factor 51/9, which product is 1,772.25 MeV.

Therefore the red charm quark constituent mass is

$$m_c = 312.75\ MeV + 1,772.25\ MeV = 2.085\ GeV$$

The blue and green charm quarks are defined similarly, and their masses are calculated similarly.

These results when added up give a total mass of second generation fermions:

$$\Sigma_{f_2} = 32.9\ GeV \tag{132}$$

8.3.2 Massless Neutrinos and Parity Violation

It is required (as an ansatz or part of the $D_4 - D_5 - E_6$ model) that the charged $W_\pm$ neutrino-electron interchange
must be symmetric with the electron-neutrino interchange,
so that the absence of right-handed neutrino particles requires
that the charged $W_\pm$ $SU(2)$ weak bosons act only on left-handed electrons.

It is also required (as an ansatz or part of the $D_4 - D_5 - E_6$ model) that each gauge boson must act consistently on the entire Dirac fermion particle sector, so that the charged $W_\pm$ $SU(2)$ weak bosons act only on left-handed fermions of all types.

Therefore, for the charged $W_\pm$ $SU(2)$ weak bosons, the 4-dimensional spinor fields $S_{8\pm}$ contain only left-handed particles and right-handed antiparticles. So, for the charged $W_\pm$ $SU(2)$ weak bosons, $S_{8\pm}$ can be denoted $S_{8\pm L}$.

The neutral W_0 weak bosons do not interchange Weyl neutrinos with Dirac fermions, and so may not entirely be restricted to left-handed spinor particle fields $S_{8\pm L}$, but may have a component that acts on the full right-handed and left-handed spinor particle fields $S_{8\pm} = S_{8\pm L} + S_{8\pm R}$.

However, the neutral W_0 weak bosons are related to the charged $W_\pm$ weak bosons by custodial $SU(2)$ symmetry, so that the left-handed component of the neutral W_0 must be equal to the left-handed (entire) component of the charged $W_\pm$.

Since the mass of the W_0 is greater than the mass of the $W_\pm$, there remains for the W_0 a component acting on the full $S_{8\pm} = S_{8\pm L} + S_{8\pm R}$ spinor particle fields.

Therefore the full W_0 neutral weak boson interaction is proportional to $(m^2_{W_\pm}/m^2_{W_0})$ acting on $S_{8\pm L}$ and
$(1 - (m^2_{W_\pm}/m^2_{W_0}))$ acting on $S_{8\pm} = S_{8\pm L} + S_{8\pm R}$.

If $(1 - (m_{W_\pm} 2/m^2_{W_0}))$ is defined to be $\sin\theta_w{}^2$ and denoted by ξ, and if the strength of the $W_\pm$ charged weak force (and of the custodial $SU(2)$ symmetry) is denoted by T,

then the W_0 neutral weak interaction can be written as:

$W_{0L} \sim T + \xi$ and $W_{0R} \sim \xi$.

The $D_4 - D_5 - E_6$ model allows calculation of the Weinberg angle θ_w, by

$$m_{W_+} = m_{W_-} = m_{W_0} \cos\theta_w \tag{133}$$

The Hopf fibration of S^3 as $S^1 \rightarrow S^3 \rightarrow S^2$ gives a decomposition of the W bosons into the neutral W_0 corresponding to S^1 and the charged pair W_+ and W_- corresponding to S^2.

The mass ratio of the sum of the masses of W_+ and W_- to the mass of W_0 should be the volume ratio of the S^2 in S^3 to the S^1 in $S3$.

The unit sphere $S^3 \subset R^4$ is normalized by $1/2$.

The unit sphere $S^2 \subset R^3$ is normalized by $1/\sqrt{3}$.

The unit sphere $S^1 \subset R^2$ is normalized by $1/\sqrt{2}$.

The ratio of the sum of the W_+ and W_- masses to the W_0 mass should then be $(2/\sqrt{3})V(S^2)/(2/\sqrt{2})V(S^1) = 1.632993$.

The sum $\Sigma_{m_W} = m_{W_+} + m_{W_-} + m_{W_0}$ has been calculated to be $v\sqrt{\alpha_w} = 517.798\sqrt{0.2534577} = 260.774\ GeV$.

Therefore, $\cos\theta_w{}^2 = m_{W_\pm}^2/m_{W_0}^2 = (1.632993/2)^2 = 0.667$, and

$\sin\theta_w{}^2 = 0.333$, so $m_{W_+} = m_{W_-} = 80.9\ GeV$, and $m_{W_0} = 98.9\ GeV$.

8.4 Corrections for m_Z and θ_w

The above values must be corrected for the fact that only part of the w_0 acts through the parity violating $SU(2)$ weak force and the rest acts through a parity conserving $U(1)$ electromagnetic type force.

In the $D_4 - D_5 - E_6$ model, the weak parity conserving $U(1)$ electromagnetic type force acts through the $U(1)$ subgroup of $SU(2)$, which is not exactly like the $D_4 - D_5 - E_6$ electromagnetic $U(1)$ with force strength $\alpha_E = 1/137.03608 = e^2$.

The W_0 mass m_{W_0} has two parts:

the parity violating $SU(2)$ part $m_{W_{0\pm}}$ that is equal to $m_{W_\pm}$; and

the parity conserving part $m_{W_{00}}$ that acts like a heavy photon.

As $m_{W_0} = 98.9$ GeV $= m_{W_{0\pm}} + m_{W_{00}}$, and as $m_{W_{0\pm}} = m_{W_\pm} = 80.9\ GeV$, we have $m_{W_{00}} = 18\ GeV$.

Denote by $\tilde{\alpha_E} = \tilde{e}^2$ the force strength of the weak parity conserving $U(1)$ electromagnetic type force that acts through the $U(1)$ subgroup of $SU(2)$.

The $D_4 - D_5 - E_6$ electromagnetic force strength $\alpha_E = e^2 = 1/137.03608$ was calculated using the volume $V(S^1)$ of an $S^1 \subset R^2$, normalized by $1/\sqrt{2}$.

The $\tilde{\alpha_E}$ force is part of the $SU(2)$ weak force whose strength $\alpha_w = w^2$ was calculated using the volume $V(S^2)$ of an $S^2 \subset R^3$, normalized by $1/\sqrt{3}$.

Also, the $D_4 - D_5 - E_6$ electromagnetic force strength $\alpha_E = e^2$ was calculated using a 4-dimensional spacetime with global structure of the 4-torus T^4 made up of four S^1 1-spheres,

while the $SU(2)$ weak force strength $\alpha_w = w^2$ was calculated using two 2-spheres $S^2 \times S^2$, each of which contains one 1-sphere of the $\tilde{\alpha_E}$ force.

Therefore $\tilde{\alpha_E} = \alpha_E(\sqrt{2}/\sqrt{3})(2/4) = \alpha_E/\sqrt{6}$,
$\tilde{e} = e/4\sqrt{6} = e/1.565$, and

the mass $m_{W_{00}}$ must be reduced to an effective value

$m_{W_{00}eff} = m_{W_{00}}/1.565 = 18/1.565 = 11.5$ GeV for the $\tilde{\alpha_E}$ force to act like an electromagnetic force in the 4-dimensional spacetime of the $D_4 - D_5 - E_6$ model:

$\tilde{e} m_{W_{00}} = e(1/5.65) m_{W_{00}} = e m_{Z_0}$,

where the physical effective neutral weak boson is denoted by Z rather than W_0.

Therefore, the correct $D_4 - D_5 - E_6$ values for weak boson masses and the Weinberg angle are:

$m_{W_+} = m_{W_-} = 80.9\ GeV$;

$m_Z = 80.9 + 11.5 = 92.4\ GeV$; and

$\sin{\theta_w}^2 = 1 - (m_{W_\pm}/m_Z)^2 = 1 - 6544.81/8537.76 = 0.233$.

Radiative corrections are not taken into account here, and may change the $D_4 - D_5 - E_6$ value somewhat.

8.5 K-M Parameters.

The following formulas use the above masses to calculate Kobayashi-Maskawa parameters:

$$phase\ angle\ \epsilon = \pi/2 \tag{134}$$

$$\sin\alpha = [m_e + 3m_d + 3m_u]/\sqrt{[m_e^2 + 3m_d^2 + 3m_u^2] + [m_\mu^2 + 3m_s^2 + 3m_c^2]} \tag{135}$$

$$\sin\beta = [m_e + 3m_d + 3m_u]/\sqrt{[m_e^2 + 3m_d^2 + 3m_u^2] + [m_\tau^2 + 3m_b^2 + 3m_t^2]} \tag{136}$$

$$\sin\tilde{\gamma} = [m_\mu + 3m_s + 3m_c]/\sqrt{[m_\tau^2 + 3m_b^2 + 3m_t^2] + [m_\mu^2 + 3m_s^2 + 3m_c^2]} \tag{137}$$

$$\sin\gamma = \sin\tilde{\gamma}\sqrt{\Sigma_{f_2}/\Sigma_{f_1}} \tag{138}$$

The resulting Kobayashi-Maskawa parameters are:

	d	s	b
u	0.975	0.222	$-0.00461i$
c	$-0.222 - 0.000191i$	$0.974 - 0.0000434i$	0.0423
t	$0.00941 - 0.00449i$	$-0.0413 - 0.00102i$	0.999

(139)

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