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TEMPERATURE, TOPOLOGY AND QUANTUM FIELDS

A Thesis Presented to The Faculty of the Division of Graduate Studies by

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Dedicated to My Parents

This thesis uses Path Integrals and Green's Functions to study

Gravity, Quantum Field Theory and Statistical Mechanics,

particularly with respect to:

finite temperature quantum systems of different spin in gravitational fields;

finite temperature interacting quantum systems in perturbative regime;

self-interacting fermi models in non-trivial space-time of different dimensions;

non-linear quantum models at finite temperatures in a background curved space-time;

3-D topological field models in non-trivial space-time and at finite temperatures;

thermal quantum systems in a background curved space-time.

Results include: NON-EQUIVALENCE of INERTIAL and GRAVITATIONAL Mass.

CONTENTS

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LIST OF FIGURES

CONVENTIONS AND ABBREVIATIONS

The sign conventions of the metric of the Part I. is $sign(+2) = diag(-, +, +, +)$ with Riemann tensor $R^{\alpha}{}_{\beta\gamma\delta} = \partial_{\delta}\Gamma^{\alpha}{}_{\beta\gamma} - \partial_{\gamma}\Gamma^{\alpha}{}_{\beta\delta} + \Gamma^{\alpha}{}_{\xi\gamma}\Gamma^{\xi}{}_{\beta\delta} - \Gamma^{\alpha}{}_{\xi\delta}\Gamma^{\xi}{}_{\beta\gamma}$ and Ricci tensor as contraction of the form $R_{\mu\nu} = R^{\alpha}{}_{\beta\gamma\alpha}$

In Part II. metric with signature $sign(-2) = diag(+, -, -, -)$ is used in order to preserve standard formulation theory in real time formalism.

The units $\hbar = c = 1$ and Boltzmann constant $k = 1$ are used in the thesis.

The following special symbols are used throughout:

$*$ complex conjugate

$+$ Hermitian conjugate

$-$ Dirac conjugate

$\partial_{\alpha} = \frac{\partial}{\partial x^{\alpha}}$ partial derivative

∇_{α} or ; covariant derivative

$\Gamma^{\delta}{}_{\beta\gamma}$ Christoffel symbols

$\omega^{\alpha}{}_{\beta\gamma}$ spin connection

tr or Tr are traces

$[a, b] = ab - ba$ commutator

$\{a, b\} = ab + ba$ anticommutator

$\sim$ order of magnitude estimate

$\simeq$ approximately equal

$\equiv$ defined to be equal to

SUMMARY

The problems which are studied in this work belong to three different fields of theoretical physics: gravity, quantum field theory and statistical mechanics. Two well known methods of modern quantum field theory, path integrals and Green's functions, allow one to connect these different branches of physics. It is possible to get a number of interesting physical predictions by combining these two methods. This combination of path integral and Green's function methods is applied to: finite temperature quantum systems of different spin in gravitational fields; finite temperature interacting quantum systems in perturbative regime; self interacting fermi models in non-trivial space-time of different dimensions; non-linear quantum models at finite temperatures in a background curved space-time; 3-D topological field models in non-trivial space-time and at finite temperatures; and construction of the statistics and thermodynamics of thermal quantum systems in a background curved space-time. The thesis is divided into four parts.

The core of the thesis is Part I. It is concerned with the development of ideas of local quantum statistics and thermodynamics of ideal thermal quantum systems. The goal is to apply the methods of quantum field theory to thermal quantum systems, and so to extend and to improve them as convenient mathematical tools for describing the thermal behavior of quantum systems of different spins in external gravitational field. Particular attention is devoted to the development of the connection between Green's functions of the models and their distribution functions. For the description

of thermal quantum systems in an external curved space-time, Schwinger-Dewitt and momentum-space representations of Green's functions are introduced. The problem of introducing temperature into field models in curved space-time for these two methods is discussed. The equivalence and difference between these two formalisms are studied. The aim of introducing these different representations is to show how to introduce chemical potential in the momentum-space formulation of partition functions of quantum systems and to study their thermal behavior in external gravitational fields. The concept of local thermodynamics is introduced, and densities of Helmholtz free energies and grand thermodynamical potentials for bose and fermi gases are computed. It is shown how average occupation numbers of bosonic and fermionic quantum system depend on external gravitational fields. Low temperature behavior of bose and fermi gases is explored, and the phenomenon of Bose condensation in thermal bose systems in a background curved space-time is studied. Thermodynamical properties of a photon gas in a background gravitational field are considered.

In part II finite temperature interacting quantum fields are explored from the point of view of thermofield dynamics. Two loop renormalizations of a self-interacting $\lambda\varphi^4$ scalar field at zero and finite temperatures are discussed and analyzed. On this basis, the two loop renormalized finite temperature Green's function of a boson and non-relativistic Hamiltonian of a boson in a heat bath in an external gravitational field are computed. The interesting problem of non-equality between inertial and gravitational masses of a boson is studied in this part.

In part III non-linear spinor models in 3-D and 4-D space-time are considered.

The Gross-Neveu model and the 3-D Heisenberg model in non-trivial space-times with different topologies are studied. The problem of the dynamical mass of composite fermions is discussed. The 4-D Heisenberg model in a background curved space-time at finite temperature is explored. The influence of curvature on the dynamical mass generation of fermions is considered.

Part IV is devoted to the analysis of topological field models in non-Euclidean 3-D space-time and finite temperature effects. The properties of vector and tensor field models with Chern-Simons mass term are considered. The effective Chern-Simons terms of vector and tensor fields appearing as the result of interaction of 3-D fermions with external gauge vector and tensor fields are discussed. The ideas and methods developed in part I are applied to the analysis of topological models with non-trivial space-time structure. The influences of the topology of 3-D space-time and of thermal behavior of interacting fermions on generation of the effective Chern-Simons terms are studied and analyzed.

New results contained in thesis

Finite temperature field theory in flat space-time has been studied in a number of journal publications (references in the text of part I) and books [Bonch-Bruevich et al 1962, Abrikosov et al. 1963, Popov 1987, Kapusta 1989 et al.]. Quantum field theory in curved space-time has been developed by many authors (journal references in the text of part I) and is well described in the books [De-Witt 1965, Birrell & Devies 1982, Fulling 1989 et al.] Finite temperature field theory in curved space-time in the Schwinger-DeWitt formalism [De-Witt 1965,1975] is considered in many journal publications (c.f. references in the text of part I). Most important are the works of Dowker, Kennedy, Denardo et al. [Dowker & Kritchley 1977, Dowker & Kennedy 1978, Denardo & Spalucci 1983]. The momentum-space method in applications to a field theory in curved space-time was discussed in the works of Bunch, Parker, Panangaden [Bunch & Parker 1979, Panangaden 1981]. This method was used in the study of the $\lambda\varphi^4$-model and the quantum electrodynamics in perturbative regimes in curved space-time. In Part I, on the basis of the above mentioned works, the author develops the theory of local thermodynamics and a statistics in curved space-time. The author shows that the thermodynamical potentials (free energy, grand thermodynamical potential) of thermal quantum systems may be computed directly through the thermal Green's functions of corresponding quantum fields in chapter III (III.24) and (III.52); chapter VII (VII.22) and chapter IX (IX.15). In chapters VII and IX the author shows that the densities of free energy of thermal bose and and fermi systems in curved space-time can be derived with the Green's function in the Schwinger-DeWitt formalism. The author shows the equivalence between the Schwinger-DeWitt and momentum space methods in applications to thermal quantum systems in chapter VII (VII.25) and chapter IX (IX.18). The author analyzes the difficulties of the computation of the grand thermodynamical potentials for bose and fermi systems using the Schwinger-DeWitt formalism (chapter IX) and computes the grand thermodynamical potentials of ideal bose and fermi systems in gravitational fields with momentum-space method in chapter XII (XII.1 and XII.2) and chapter XIII. The author considers low temperature properties of these systems in chapters

XII and XIII, and studies the behavior of the chemical potentials of the bose and fermi systems with respect to space-time curvature (XII.19) and (XIII.20). The author analyzes the phenomenon of Bose condensation (section XII.3) and finds the variation of the critical temperature of condensation with respect to the variation of the curvature (XII.25). The author studies the thermodynamical properties of a photon gas in curved space-time with momentum space-methods in chapter X. The author computes Bose-Einstein and Fermi-Dirac distributions in curved space-time in chapters XII (XII.14) and XIII (XIII.10)

The temperature properties of self-interacting and gauge fields are studied in imaginary and in real time formalisms in the works of Kislinger , Kapusta, Donoghue et al., from the point of view of their renormalizability [Kislinger et al., 1976] and finite temperature behavior of the constant of the interaction [Kapusta 1979, Fujimoto et al., 1987]. In the work of Donoghue the gravitational field was also introduced in the electrodynamics with the Tolman's shift of temperature, and the non-equality between inertial and gravitational masses of fermion was also discussed [Donoghue et al.,1984]. In part II of this work the author computes the finite temperature effective mass and renormalized Green's function of boson in a heat bath in two loop approximation with a real time formalism in chapter XV (XV.21), finds the non-relativistic finite temperature Hamiltonian of bosons in a weak gravitational field (section XVI.1), and gets the ratio between the inertial and the gravitational masses of a boson (section XVI.2) (XVI.9).

Twisted fields were introduced by Isham in 1978 and field models in space-time with a non-Minkowskian topology were studied in the works of Ford, Toms et al., [Ford 1980, Toms 1980]. Non-linear 3-D field models with γ_5 symmetry and a large number of flavors and $(\bar{\psi}\psi)^2$ non-linear models in 3-D and 4-D space-time were studied in works Gross & Neveu 1974, Bender 1977, Tamvakis, Kawati & Miyata 1981 et al. In part III the author applies the idea of twisted and untwisted spinor fields to obtain the dependence of the dynamical fermionic mass with respect to a non-trivial topology of space-time in chapter XVIII (section XVIII.2). The author analizes the behavior of the dynamic fermionic mass with respect to the curvature of space-time and the thermal behavior of interacting fields (section XVIII.4).

The idea of computing the induced Chern-Simons action for vector and tensor fields as the result of the interaction of 3-D fermi fields with external vector and tensor fields was studied in the works of Redlich, Das, Ojima et al. Das proposed the method of the computation of finite temperature induced Chern-Simons action of vector type with the method of derivative expansion [Babu & Das 1987]. Ojima computed the Chern-Simons action of vector and tensor fields using the path integral method [Ojima 1989]. In Part IV the author, using the above mentioned works computes the induced Chern-Simons mass term of a vector field in space-times with topologies $R^2 \times S^1$ and $R^2 \times mobius\ strip$ in section (XX.2). In chapter XXI the author computes the finite temperature gravitational induced Chern-Simons mass term using the momentum space method developed in part I.

In the list of references the author includes only the publications most important for undestanding the text.

PART I

LOCAL QUANTUM STATISTICS AND THERMODYNAMICS IN CURVED SPACE-TIME

Introduction

The statistics and thermodynamics of quantum systems in gravitational fields have the attracted attention of physicists for a long time. The equilibrium distribution of a relativistic quantum gas in a gravitational field has been studied using the kinetic Boltzmann equation, and it was found that the solution of a functional Boltzmann equation is a relativistic Maxwell distribution for a certain type of gravitational field [Chernikov 1962]. Later, different variants of non-equilibrium statistical mechanics in classical general relativity as a generalization of the standard statistical equations for classical systems in curved space-time were proposed [Israel & Kandrup 1984, Ignat'ev 1985].

At the same time field-theoretical methods were developed and applied to statistical mechanics and thermodynamics. Since the middle of the 50's, significant progress was made in quantum theory of many body systems, and was connected with the development of finite temperature quantum field methods in statistical physics. Feynman [Feynman R.1953] studied the λ-transition in Helium using the partition function in the form of a path integral in quantum mechanics. Martin and Schwinger [Martin & Schwinger 1959] considered a many-particle system in the context of quantum field theory in order to treat multiparticle system from the quantum field-theoretical point of view. They described the microscopic behavior of a multiparticle system using Green's functions, and found that the finite temperature Green's functions are related to intensive macroscopic variables when the energy and number of particles are large. Matsubara [Matsubara 1955] proposed finite temperature perturbation techniques similar to perturbative quantum field theory and calculated the grand partition function by introducing Green's functions in imaginary time formalism. The works of Feynman, Matsubara, and Martin & Schwinger created the basis for understanding the close connection between Euclidean field theory and statistical mechanics.

In the 60's convenient methods using finite temperature calculations for interacting thermal systems with a propagator formalism in the perturbative regime were proposed in [Bonch-Bruevich & Tyablikov 1962], [Abrikosov et al. 1963], [Fradkin 1965], and [Symanzik 1966]. These works stressed the analogy between Euclidean Green functions and distribution functions in statistical mechanics. Analogies between statistical physics and quantum field theory at finite temperatures were further strengthened in the study of infinite equilibrium systems of scalar and spinor fields [Dolan & Jackiw 1974]. These works provided an opportunity to construct the statistical mechanics and thermodynamics of quantum systems in the language of finite temperature field theory, and pointed the way towards a construction of a statistical mechanics and thermodynamics in curved space-time.

In the 70's, interest in quantum statistical processes in general relativity was aroused by the works of Hawking [Hawking 1975], and Bekenstein [Bekenstein 1973] on the thermodynamics of black holes, where an intimate connection between thermodynamics and the structure of space-time was pointed out.

Parallel to the research on the thermodynamics of black holes, quantum effects present in the early Universe were thoroughly discussed by Guth et al. [Guth 1966, Starobinsky 1982, Linde 1984]. From these works one may conclude that, for early time and high temperatures, the quantum statistical properties of matter and strong curvature of space-time become significant.

With the development of a quantum field theory using functional integral methods at finite temperature [Weinberg 1974, Faddeev & Slavnov 1991], [Kapusta 1989], [Popov 1987] and of a formalism of a quantum theory of field in curved space-time [DeWitt 1965, Fulling 1989], new possibilities for studing the behaviour of thermal systems with non-trivial geometry and topology were discussed. This approach was developed by Denardo et al. [Denardo & Spalucci 1983], [Nakazava & Fukuyama 1985].

Hu [Hu 1981-3] introduced time dependent temperature $T(t) = a(t)/T$ for an equilibrium gas of scalar particles which is described by a conformal scalar field in a closed Robertson-Walker (RW) Universe in a different way, by using the equilibrium temperature of a flat space-time T.

The problems of a thermodynamical description of a non-equilibrium thermal quantum gas and conditions for thermal equilibrium in an expanding Universe were studied by Hu [Hu 1982]. However, in general this problem has not been solved satisfactorily. Difficulties in the construction of a thermodynamics of thermal systems in dynamical space-time are connected with such problems as the definitions of temperature, energy spectrum, and the vacuum state of a thermal system.

In Part I of this work we extend the results of finite temperature field theory in order to construct a statistical mechanics and thermodynamics of bosonic and fermionic quantum systems in an external curved space-time.

The methods developed here are based on using the language of a finite temperature Green's function in an arbitrary curved space time for the definition of the thermodynamical potentials of thermal quantum systems.

Green's functions of matter fields in curved space-time are nonlocal objects that are well defined in a small range of the space-time manifold [DeWitt 1965] [Nakazava & Fukuyama 1985]. Thermal properties of quantum systems may be considered the properties of the set of quasi-equilibrium sub-systems, which are elements of the whole quantum system [Kulikov & Pronin 1993]. This suggestion will lead us to the definition of distribution functions of bose and fermi systems, and some interesting thermodynamical consequences.

The division of the entire system into quasi-equilibrium sub-systems may be done in a simple way using Riemann normal coordinates [Petrov 1969]. These coordinates permit us to rewrite Green's functions in a momentum space representation, and to introduce a local temperature in order to write thermodynamical potentials through these Green's functions. As a result, the thermodynamical potentials will be written as a series expansion in powers of the curvature tensor at a selected point of the curved space-time manifold. Therefore, at any point in space-time the coefficients of the series will change because the curvature will change for all other points on the manifold.

Therefore, in accordance with our suggestions, densities of thermodynamic potentials will be functionals of curvature, and they will be directly connected to the temperature Green's functions by a simple relation. The aim of this part will be to develop the mathematical formalism of temperature Green's functions in external curved space-time and to use the formalism to construct a quantum statistics and thermodynamics of thermal bosonic and fermionic ideal quantum systems and a thermodynamics of a photon gas in an arbitrary curved space-time.

Part I is organized in the following way: A short review of statistical mechanics is presented in chapter I. In chapter II functional integral methods are applied to non-relativistic and relativistic many-body systems in Euclidean space-time. The purpose will be to generalize these methods for a description of statistical systems in curved space-time. Finite temperature Green's functions are introduced in chapter III for computation of the thermodynamical potentials of quantum gases. Finite temperature gauge fields are studied in chapter IV. An introduction to quantum fields in curved space-time is presented in chapter V. The bosons and fermions in external gravitational fields are studied in chapters VI and VIII. The thermodynamics of bose and fermi gases in curved space-time is studied in chapters VII and IX. The thermodynamics of vector bosons is considered in chapter X. Renormalization problems are considered in chapter XI. The phenomenon of Bose-Einstein condensation and the low temperature properties of fermi gas are studied in chapters XII and XIII.

Chapter 1

QUANTUM FIELD METHODS

IN STATISTICAL PHYSICS

The customary approach to many-body theory is the method of second quantization [Abrikosov et al. 1963, Fetter & Walecka 1971]. This chapter is a short introduction to statistical mechanics of simple quantum systems using this method.

1.1 Equilibrium statistical mechanics

Statistical mechanics deals with three types of ensembles:

1) *The microcanonical ensemble* is used to describe an isolated system which has a fixed energy E, particle number N, and volume V.

2) *The canonical ensemble* is used to describe a system in contact with a heat reservoir at temperature T. The system can freely exchange energy with the reservoir

and T, N, and V are fixed variables

3) *The grand canonical ensemble* is used to describe a system which can exchange particles as well as energy with reservoir. In this ensemble T, V and the chemical potential μ are fixed variables.

The main aim of statistical mechanics is to derive the thermodynamic properties of macroscopic bodies starting from the description of the motion of the microscopic components (atoms, electrons, etc.).

To solve the problem it is necessary to find the probability distribution of the microscopic components in thermal equilibrium (after a sufficiently long time), and deduce the macroscopic properties of the system from this microscopic probability distribution.

Following this scheme let us consider a classical Hamilton system with $2N$ degrees of freedom in a box of volume V. The classical equations of motion are:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} \tag{1.1}$$

where H is the Hamiltonian, q_i and p_i $(i = 1, 2, ..., N)$ are the set of coordinates and momenta of the system.

Let $A(q, p)$ be an arbitrary measurable quantity (an observable). The equilibrium average of this quantity $\bar{A}$ is defined by

$$\bar{A} = \lim_{t\to\infty} \frac{1}{t} \int_o^\infty A(q(\tau), p(\tau)) d\tau$$

$$= \int dqdpA(q,p)\varrho(q,p) \tag{1.2}$$

where $\varrho(q,p)$ is the equilibrium probability.

This function is never negative and satisfies the normalization condition

$$\int dqdp \ \varrho(q,p) = 1. \tag{1.3}$$

The fundamental hypothesis of equilibrium statistical mechanics is that ϱ follows the canonical distribution

$$\varrho(q,p) = Z^{-1} \exp[-\beta H(q,p)], \tag{1.4}$$

where

$$Z = \int dqdp \exp[-\beta H(q,p)] \tag{1.5}$$

is the partition function, and $\beta^{-1} = T$ is the absolute equilibrium temperature.

It is useful to find a characterization of the canonical distribution that distinguishes it from all other possible probability distributions. It is convenient to introduce the entropy of a distribution ϱ defined as follows:

$$S[\varrho] = -\int dqdp\varrho(q,p)\ln\varrho(q,p) = -<\ln\varrho> \tag{1.6}$$

Entropy has the following properties: the more ordered is the system the smaller is the entropy (i.e., the more concentrated the probability distribution in a restricted region of phase space); the more disordered is the system (i.e., the more uniform the probability distribution), the larger is the entropy.

For relativistic quantum statistical systems the equation (1.4) will be turned into the equation for statistical operator

$$\hat{\varrho} = Z^{-1} \exp[-\beta(\hat{H} - \mu\hat{N}] \tag{1.7}$$

where $\hat{H}$ is the Hamiltonian and $\hat{N}$ is the number operator. This operator is Hermitian and commutes with Hamiltonian H. The parameter μ is the so called chemical potential.

The ensemble average of an operator $\hat{A}$ will be

$$< A >= Z^{-1}\mathrm{Tr}[\hat{A}\hat{\varrho}] \tag{1.8}$$

The factor Z will turn into a so called grand canonical partition function of the form

$$Z = \mathrm{Tr}\exp[-\beta(\hat{H} - \mu\hat{N}] \tag{1.9}$$

This function is the most important in thermodynamics.

The average value of the energy U may be written from (1.8) as

$$U =< \hat{H} >= \mathrm{Tr}[\hat{H}\hat{\varrho}] \tag{1.10}$$

and the entropy as

$$S = -\mathrm{Tr}[\hat{\varrho}\ln\hat{\varrho}] \tag{1.11}$$

From the equations (1.9) and (1.11) we get a useful relation

$$S = -\mathrm{Tr}[\hat{\varrho}\ \ln\hat{\varrho}]$$

$$= -\mathrm{Tr}[\hat{\varrho}(-\ln Z - \beta\hat{H} + \mu\hat{N})]$$

$$= \ln Z + \beta U - \beta\mu N,$$

or

$$(1/\beta)\ln Z = U - S/\beta - \mu N. \tag{1.12}$$

The quantity

$$\Omega = -(1/\beta)\ln Z \tag{1.13}$$

is the grand thermodynamical potential of the grand canonical ensemble.

For canonical ensemble ($\mu = 0$) we can write the equivalent equation for thermodynamic potential F (free energy).

It is easy to find that

$$F - \Omega = \mu N. \tag{1.14}$$

The grand partition function $Z = Z(V, T, \mu)$ is the most important function in thermodynamics. All other standard termodynamic properties may be determined from this function. For example, the pressure, particle number, entropy, and energy (in the infinite volume limit) are

$$P = T\frac{\partial \mathrm{ln} Z}{\partial V}, \qquad N = T\frac{\partial \mathrm{ln}\, Z}{\partial \mu}$$

$$S = T\frac{\partial\ T\mathrm{ln} Z}{\partial T}, \qquad E = -PV + TS + \mu N \tag{1.15}$$

1.2 Statistical mechanics of simple systems Formalism of second quantization

Now we can apply the formalism of statistical mechanics developed above to non-interacting many body quantum systems in the frame of the method of second quantization.

1. Bosonic quantum system at finite temperature

Let us study a bosonic quantum system. Each quantum state ϵ of the system may be occupied by bosons. Let n ($n = 0, 1, 2, 3, ...$) be number of bosons in this state, and

$|n>$ be the wave function of this state. We will call a state $|0>$ a vacuum state. One may introduce creation a and annihilation a^+ operators with commutation relation

$$[a, a^+] = aa^+ - a^+a = 1 \tag{1.16}$$

The action of these operators on eigenstates is

$$a|n>= \sqrt{n}|n-1>, \qquad a^+|n>= \sqrt{n+1}|n+1> \tag{1.17}$$

and on vacuum state is

$$a|0>= 0 \tag{1.18}$$

From (1.16), (1.17) one can get the number operator $\hat{N} = a^+a$:

$$\hat{N} = a^+a|n>= n|n> \tag{1.19}$$

We may build all states $|n>$ from vacuum $|0>$ by repeated applications of the creation operator

$$|n>= (n!)^{-1/2}(a^+)^n|0> \tag{1.20}$$

The Hamiltonian of the system in the state with energy ϵ may be constructed as a product ϵ with number operator (up to an additive constant) in the form

$$\hat{H} = \epsilon(\hat{N} + 1/2) = \epsilon(a^+a + 1/2) = \frac{1}{2}\epsilon(a^+a + aa^+) \tag{1.21}$$

The additive term $\epsilon/2$ in (1.21) is the zero-point energy. Usually this term may be ignored and we get $\hat{H} = \epsilon\hat{N}$.

The grand partition function according to (1.9) will be

$$Z = \mathrm{Tr}\exp[-\beta(\hat{H} - \mu\hat{N})] = \mathrm{Tr}\exp[-\beta(\epsilon - \mu)\hat{N}]$$

$$\sum_{n=0}^{\infty} < n|\exp[-\beta(\epsilon - \mu)\hat{N}]|n >= \sum_{n=0}^{\infty} \exp[-\beta(\epsilon - \mu)n]$$

$$= (1 - \exp[-\beta(\epsilon - \mu)])^{-1} \tag{1.22}$$

Inserting (1.22) into (1.15) we get the average number of bosons in the system with energy ϵ

$$N = (\exp[\beta(\epsilon - \mu)] - 1)^{-1} \tag{1.23}$$

and the average energy in the form $\bar{\epsilon} = \epsilon N$.

2. Fermionic quantum system at finite temperature

For a fermionic quantum system there are only two states $|0>$ and $|1>$ with energy ϵ.

The action of the fermion creation and annihilation operators on these states is

$$\alpha^+|0>=|1>,\quad \alpha|1>=|0>$$

and

$$\alpha^+|1>=|0>,\quad \alpha|0>=|1> \tag{1.24}$$

Thus, these operators have the property that their square is zero

$$\alpha^+\alpha^+=\alpha\alpha=0 \tag{1.25}$$

The number operator is $\hat{N}=\alpha^+\alpha$. One has

$$\hat{N}|0>=\alpha^+\alpha|0>=0;\quad \hat{N}|1>=\alpha^+\alpha|1>=|1> \tag{1.26}$$

Creation and annihilation operators satisfy the anticommutation relation

$$\{\alpha,\alpha^+\}=\alpha\alpha^++\alpha\alpha^+=1 \tag{1.27}$$

The Hamiltonian for this system may be taken in the form

$$\hat{H}=\frac{1}{2}(\alpha^+\alpha-\alpha\alpha^+)=\epsilon(\hat{N}-1/2) \tag{1.28}$$

The partition function for the fermionic system will be

$$Z = \mathrm{Tr}\exp[-\beta(\hat{H} - \mu\hat{N})] = \sum_{n=0}^{1} < n|\exp[-\beta(\epsilon - \mu)\hat{N}]|n >$$

$$= 1 + \exp[-\beta(\epsilon - \mu)] \tag{1.29}$$

The average number of fermions is

$$N = (\exp[\beta(\epsilon - \mu)] + 1)^{-1} \tag{1.30}$$

The average energy of the system in the state with energy ϵ is $\bar{\epsilon} = \epsilon N$.

The next section describes an alternative approach, the method of functional integrals for studying the behavior of statistical systems.

Chapter 2

PATH INTEGRALS

IN STATISTICAL PHYSICS

Functional integration, introduced several decades ago [Feynman & Hibbs 1965], is one of the most powerful methods of modern theoretical physics. The functional integration approach to systems with an infinite number of degrees of freedom turns out to be very suitable for the introduction and formulation of perturbation theory in quantum field theory and statistical physics. This approach is simpler than the operator method.

Using functional integrals in statistical physics allows one to derive numerous interesting results more quickly than other methods. Theories of phase transitions of the second kind, superfluidity, superconductivity, plasma, and the Ising model are examples of the problems for which the functional integration method appears to be very useful. If an exact solution exists, the functional integration method gives a simple way to obtain it. In other cases, when exact knowledge is unobtainable, the application of functional integrals helps to build a qualitative picture of the phenomenon, and to develop an approximate solution scheme.

The functional integration method is suitable for obtaining the diagram techniques of the perturbation theory, and also for modifying the perturbative scheme if such a modification is necessary. The extension of functional integral techniques to background curved space-time allows one to take into account the gravitational field by considering the statistical and thermodynamical properties of the systems.

The aim of this chapter is to introduce the finite temperature functional integral approach to statistical mechanics and local thermodynamics in curved space-time.

2.1 Partition function in path integral formalism

This paragraph discusses the scalar field which is described by the Schrodinger field operator $\hat{\varphi}(\vec{x})$ where $\vec{x}$ is the spatial coordinate. We denote eigenstates of $\hat{\varphi}(\vec{x})$ by $|\varphi>$

$$\hat{\varphi}(\vec{x})|\varphi>=\varphi(\vec{x})|\varphi> \tag{2.1}$$

where $\varphi(\vec{x})$ is a c-number function. Let $\hat{\pi}(\vec{x})$ be its conjugate momentum operator.

The usual completeness and orthogonality conditions may be written as:

$$\int d\varphi\,|\varphi\rangle\,\langle\varphi| = 1$$

$$\langle \varphi_a \mid \varphi_b \rangle = \delta \left[\varphi_a(\vec{x}) - \varphi_b(\vec{x}) \right] \tag{2.2}$$

where δ-is the Dirac delta function.

Similarly, the eigenstates of the conjugate momentum field operator are labeled by $|\pi >$ and satisfy the equation

$$\hat{\pi}(\vec{x})|\pi >= \pi(\vec{x})|\pi > \tag{2.3}$$

with eigenvalies $\pi(\vec{x})$. The completeness and orthogonality conditions are

$$\int d\pi \left|\pi\right\rangle \left\langle\pi\right| = 1$$

$$\langle \pi_a \mid \pi_b \rangle = \delta \left[\pi_a(\vec{x}) - \pi_b(\vec{x}) \right] \tag{2.4}$$

the overlap between an eigenstate of field operator and an eigenstate of the momentum operator is

$$\langle \varphi \mid \pi \rangle = \exp\left[i \int d^3x \pi(\vec{x}) \varphi(\vec{x}) \right] \tag{2.5}$$

For dynamics, one requires a Hamiltonian expressed as a functional of the field and its conjugate momentum:

$$H = \int d^3x H(\hat{\pi}, \hat{\varphi}) \tag{2.6}$$

Suppose that a system is in state $|\varphi_a>$ at a time $t=0$. After time t_f it will evolve into $exp(-iHt_f)|\varphi_a>$. The transition amplitude for going from state $|\varphi_a>$ to some other state $|\varphi_b>$ after time t_f is thus

$$< \varphi_b|\exp(-iHt_f)|\varphi_a > \tag{2.7}$$

For statistical mechanics purposes, consider the case in which the system returns to its original state after time t_f. Divide the time interval $[0,t_f]$ into N equal steps of duration $\Delta t = t_f/N$. Then

$$\langle\varphi_a|\exp(-iHt_f)|\varphi_a\rangle = \lim_{N\to\infty}\int\prod_{i=1}^{N}\left(d\varphi_i\frac{d\pi_i}{2\pi}\right)$$

$$\langle\varphi_a \mid \pi_N\rangle\langle\pi_N|\exp(-iH\Delta t_f)|\varphi_N\rangle\langle\varphi_N \mid \pi_{N-1}\rangle\langle\pi_{N-1}|\exp(-iH\Delta t_f)|\varphi_{N-1}\rangle \times ...$$

$$\times\langle\varphi_2 \mid \pi_1\rangle\langle\pi_1|\exp(-iH\Delta t_f)|\varphi_1\rangle\langle\varphi_1 \mid \varphi_a\rangle \tag{2.8}$$

It is known, that

$$< \varphi_1|\varphi_a >= \delta(\varphi_1 - \varphi_a) \tag{2.9}$$

and

$$< \varphi_{i+1}|\pi_i >= exp\left[i\int d^3x\pi_i(\vec{x})\varphi_{i+1}(\vec{x})\right] \tag{2.10}$$

For $\Delta t \to 0$ one can expand

$$\langle \pi_i | \exp(-iH\Delta t) | \varphi_i \rangle \approx \langle \pi_i | (1 - iH\Delta t) | \varphi_i \rangle =$$

$$\langle \pi_i \mid \varphi_i \rangle (1 - iH_i\Delta t) = (1 - iH_i\Delta t) \exp\left[-i \int d^3x \pi_i(x)\varphi_i(x)\right] \tag{2.11}$$

where $H_i = H(\pi_i, \varphi_i)$. Inserting (2.9), (2.10) and (2.11) into (2.8) gives

$$\langle \varphi_a | \exp(-iHt_f) | \varphi_a \rangle = \lim_{N\to\infty} \int \prod_{i=1}^{N} \left(d\varphi_i \frac{d\pi_i}{2\pi} \right) \delta(\varphi_1 - \varphi_a) \times$$

$$\exp\left\{ -i\Delta t \sum_{j=1}^{N} \int d^3x \left[H(\pi_j, \varphi_j) - \pi_j(\varphi_{j+1} - \varphi_j)\Delta t \right] \right\} \tag{2.12}$$

where $\varphi_{N+1} = \varphi_1 = \varphi_a$.

Taking the continuum limit of (2.12) , gives the result

$$\langle \varphi_a | \exp(-iHt_f) | \varphi_a \rangle = \int D\pi \int\limits_{\varphi(\vec{x},0)=\varphi_a(\vec{x})}^{\varphi(\vec{x},t_f)=\varphi_a(\vec{x})} D\varphi \times$$

$$\exp\left\{ i \int\limits_0^{t_f} dt \int d^3x \left[\pi(\vec{x}, t) \frac{\partial \varphi(\vec{x}, t)}{\partial t} - H(\pi(\vec{x}, t), \varphi(\vec{x}, t)) \right] \right\} \tag{2.13}$$

The integration over $\pi(\vec{x}, t)$ is unrestricted, and integration over $\varphi(\vec{x}, t)$ is such that the field starts at $\varphi_a(\vec{x})$ at $t = 0$ and ends at $\varphi_a(\vec{x})$ at $t = t_f$. Note that all operators are gone and r one can work only with classical variables [Kapusta 1989].

2.2 Partition function for bosons.

The partition function in statistical mechanics is expressed as

$$Z_\beta = \operatorname{Tr} \exp\left\{-\beta\left(H - \mu N\right)\right\} =$$

$$\int d\varphi_a \left\langle \varphi_a \right| \exp\left\{-\beta\left(H - \mu N\right)\right\} \left|\varphi_a\right\rangle \tag{2.14}$$

where the sum runs over all states.

The aim now is to express Z_β in terms of a functional integral.

First introduce the imaginary time $\tau = it$. The limits of integration on τ are $[0, \beta]$, then $-it_f = \beta$ and

$$Z_\beta = \int D\pi \int\limits_{periodic} D\varphi \times$$

$$\exp\left\{\int\limits_0^\beta d\tau \int d^3x \left[i\pi(x,t)\frac{\partial\varphi(x,t)}{\partial\tau} - H(\pi,\varphi) + \mu N(\pi,\varphi)\right]\right\} \tag{2.15}$$

Integration over the field is constrained so that

$$\varphi(\vec{x}, 0) = \varphi(\vec{x}, \beta) \tag{2.16}$$

This is a consequence of the trace operation.

In the equation (2.15) make the replacement

$$H(\pi,\varphi) \to H(\pi,\varphi) - \mu N(\pi,\varphi) \tag{2.17}$$

where $N(\pi,\varphi)$ is conserved charge density, if the system admits some conserved charge.

Rewrite the equation (2.15) in a more convenient form.

For this purpose, consider a scalar field, which is described by the Lagrangian of the form

$$L = -\frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m^2\varphi^2 - U(\varphi) \tag{2.18}$$

where $U(\varphi)$ is a potential function.

The momentum conjugate to the field is

$$\pi = \frac{\partial L}{\partial(\partial_0\varphi)} = \partial_0\varphi \tag{2.19}$$

Here the Hamiltonian may be written as

$$H = \pi\partial_0\varphi - L =$$

$$\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\varphi)^2 + \frac{1}{2}m^2\varphi^2 + U(\varphi) \tag{2.20}$$

For evaluating the partition function, return to the discretized version of (2.15):

$$Z_\beta = \lim_{N\to\infty} \left(\prod_{i=1}^{N} \int_{-\infty}^{\infty} \frac{d\pi_i}{2\pi} \int_{periodic} d\varphi_i \right) \exp\left\{ \sum_{j=1}^{N} \int d^3x \, [i\pi_j \, (\varphi_{j+1} - \varphi_j) - \right.$$

$$\left. -\Delta\tau \left(\frac{1}{2}\pi_j^2 + \frac{1}{2}(\nabla\varphi_j)^2 + \frac{1}{2}m^2\varphi_j^2 + U(\varphi) \right) \Big] \right\} \tag{2.21}$$

Using the equation for a Gaussian integral

$$\frac{1}{\sqrt{2\pi i}} \int_{-\infty}^{\infty} dx \exp\left\{ \frac{i}{2}ax^2 + ibx \right\} = \frac{1}{\sqrt{a}} \exp\left\{ -\frac{i}{2}\frac{b^2}{a} \right\} \tag{2.22}$$

one can write

$$\int \frac{d\pi_j}{2\pi} \exp\left\{ i\pi_j(\varphi_{j+1} - \varphi_j) - \frac{\Delta\tau}{2}\pi_j^2 \right\}$$

$$= \frac{1}{\sqrt{2\pi\Delta\tau}} \exp\left\{ -\frac{(\varphi_{j+1} - \varphi_j)^2}{\Delta\tau} \right\} \tag{2.23}$$

and after momentum integrations in (2.21) one gets

$$Z_\beta = N' \lim_{N\to\infty} \int \left(\prod_{i=1}^{N} d\varphi_i \right) \exp\left\{ \Delta\tau \sum_{j=1}^{N} \int d^3x \left[-\frac{1}{2} \left(\frac{(\varphi_{j+1} - \varphi_j)}{\Delta\tau} \right)^2 - \right.\right.$$

$$\left.\left. -\frac{1}{2}(\nabla\varphi_j)^2 - \frac{1}{2}m^2\varphi_j^2 - U(\varphi) \right] \right\} \tag{2.24}$$

Returning to the continuum limit, one obtains:

$$Z_\beta = N' \int_{periodic} D\varphi \exp\left(\int_0^\beta d\tau \int d^3x L(\varphi, \partial\varphi) \right) \tag{2.25}$$

Equation (2.25) expresses Z_β as a functional integral in time interval $[0, \beta]$. The normalization constant N' is irrelevant, since it does not change the thermodynamics.

Now it is seen how this method works in application to a non-interacting scalar field. Let the Lagrangian of the scalar field be

$$L(\varphi, \partial\varphi) = -\frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m^2\varphi^2 \tag{2.26}$$

Define the finite temperature action of the Bose system at temperatute T in a large cubic volume $V = L^3$ as

$$S_\beta = -\frac{1}{2}\int_0^\beta d\tau \int_V d^3x\varphi \left(\partial_\tau^2 + \partial_i^2 + m^2\right)\varphi \tag{2.27}$$

Due to the constraint of periodicity of the field $\varphi(\vec{x}, \tau)$ $(\vec{x} \in V, \tau \in [0, \beta])$ (2.16), it can be expanded in a Fourier series as

$$\varphi(\vec{x}, \tau) = (\beta V)^{-1/2} \sum_{n=-\infty}^{\infty} \sum_{\vec{k}} \varphi_n(\vec{k}) \exp i(\vec{k}\vec{x} + \omega_n \tau) \tag{2.28}$$

where $\varphi_n(\vec{k})$ is the Fourier coefficient, $\omega_n = 2\pi n/\beta$, and $\vec{k} = 2\pi m/L$ n, m are integer numbers.

Using the identity

$$\int_0^\beta d\tau \exp i(\omega_n - \omega_m)\tau = \beta\delta_{n,m} \tag{2.29}$$

one obtains the equation for the action in terms of the Fourier coefficients

$$S_\beta = -\frac{1}{2}\sum_{n=-\infty}^{\infty}\sum_{\vec{k}}\varphi_n^*(\vec{k})(\omega_n^2 + \vec{k}^2 + m^2)\varphi_n(\vec{k}) \tag{2.30}$$

where $\varphi_{-n}(-\vec{k}) = \varphi_n^*(\vec{k})$ goes from the reality of the field φ.

Then the partition function (2.25) may be written as

$$Z_\beta = N' \int \prod_n \prod_{\vec{k}} d\varphi_n(\vec{k}) \exp\left\{-\frac{1}{2}\varphi_n^*(k)(\omega_n^2 + \vec{k}^2 + m^2)d\varphi_n(k)\right\} \tag{2.31}$$

where the explicit form of the measure of the functional integration (2.31) is

$$D\varphi = \prod_n \prod_{\vec{k}} d\varphi_n(\vec{k}) \tag{2.32}$$

So, integration over the coefficients $\varphi_n(\vec{k})$ in (2.31) gives the following equation for the logarithm of the partition function [Bernard 1974]

$$\ln Z_\beta = -\frac{1}{2}\ln Det(\omega_n^2 + \epsilon_{\vec{k}}^2) = -\frac{1}{2}\sum_n \sum_{\vec{k}} \ln(\omega_n^2 + \epsilon_{\vec{k}}^2) \tag{2.33}$$

where $\epsilon_{\vec{k}}^2 = \vec{k}^2 + m^2$ is the energy of a one particle state with a certain momentum. Summation in respect with ω_n in (2.33) may be done with the help of the basic equation

$$\sum_{n=-\infty}^{\infty} \frac{z}{z^2 + n^2} = \frac{\pi}{2}\coth \pi z \tag{2.34}$$

The derivative of the sum gives

$$\frac{\partial}{\partial \epsilon}\sum_n \ln\left(\frac{4\pi^2 n^2}{\beta^2}+\epsilon^2\right) = \sum_n \frac{2\epsilon}{4\pi^2 n^2/\beta^2+\epsilon^2} =$$

$$= 2\beta\left(\frac{1}{2}+\frac{1}{\exp\beta\epsilon - 1}\right) \tag{2.35}$$

After integration with respect to ϵ we will have

$$\sum_n \ln\left(\frac{4\pi^2 n^2}{\beta^2}+\epsilon^2\right) = 2\beta\left[\frac{\epsilon}{2}+\frac{1}{\beta}\ln(1-\exp[-\beta\epsilon])\right] \tag{2.36}$$

The expression for the thermodynamic potential will have the standard form

$$F = -\frac{1}{\beta}\ln Z_\beta = \frac{1}{\beta}\sum_{\vec{k}} \ln(1-\exp[-\beta\epsilon_{\vec{k}}]) \tag{2.37}$$

Here the infinite, temperature independent part of the function Z_β is dropped

2.3 Green's function of boson field

Introduce the two point Green function (or propagator) of boson field at finite temperature as the thermal average of a time-ordered product of two scalar field operators[1].

$$G_\beta(x-y) =< \mathrm{T}\varphi(x)\varphi(y) >_\beta$$

[1] In quantum field theory two-points Green's function is introduced as the vacuum expectation value of a time-ordered product of two field operators.

$$= \mathrm{Tr}\left[\exp(-\beta H)\mathrm{T}\varphi(x)\varphi(y)\right]/\mathrm{Tr}\exp(-\beta H) \tag{2.38}$$

One may express the Green's function as the sum

$$G_\beta(x-x^{'}) = \theta(\tau-\tau^{'})G^+_\beta(x-x^{'}) + \theta(\tau^{'}-\tau)G^-_\beta(x-x^{'}) \tag{2.39}$$

where $G^{\pm}$ are Wightman functions

$$G^+_\beta(x-x^{'}) =< \mathrm{T}\varphi(x)\varphi(x^{'}) >_\beta$$

$$G^-_\beta(x-x^{'}) =< \mathrm{T}\varphi(x^{'})\varphi(x) >_\beta \tag{2.40}$$

and

$$\theta(\tau) = \begin{cases} 1, \tau > 0, \\ 0, \tau < 0. \end{cases} \tag{2.41}$$

is the step function.

In the interval $[0, \beta]$

$$G_\beta(x-y)_{|x^4=0} = G^+_\beta(x-y)_{|x^4=0}$$

and

$$G_\beta(x-y)_{|x^4=\beta} = G_\beta^-(x-y)_{|x^4=\beta} \tag{2.42}$$

Using the fact that $exp(-\beta H)$ and time ordering operation commute and taking into account the cyclic property of the trace, one gets an important property of the thermal Green's function

$$G_\beta(x-y)_{|x^4=0} = G_\beta^+(x-y)_{|x^4=0}$$

$$= \mathrm{Tr}\left[\exp(-\beta H)\varphi(\vec{x},\tau)\varphi(\vec{y},0)\right]/\mathrm{Tr}\exp(-\beta H)$$

$$= \mathrm{Tr}\left[\varphi(\vec{y},0)\exp(-\beta H)\varphi(\vec{x},\tau)\right]/\mathrm{Tr}\exp(-\beta H)$$

$$= \mathrm{Tr}\left[\exp(-\beta H)(\exp(\beta H)\varphi(\vec{y},0)\exp(-\beta H))\varphi(\vec{x},\tau)\right]/\mathrm{Tr}\exp(-\beta H)$$

$$= \mathrm{Tr}\left[\exp(-\beta H)\varphi(\vec{y},\beta)\varphi(\vec{x},\tau)\right]/\mathrm{Tr}\exp(-\beta H)$$

$$= G_\beta^-(x-y)_{|x^4=\beta} = G_\beta(x-y)_{|x^4=\beta} \tag{2.43}$$

Thus one has a periodicity condition

$$G_\beta(x-y)_{|x^4=0} = G_\beta(x-y)_{|x^4=\beta} \tag{2.44}$$

This relation leads to the Green's function in Euclidean quantum field theory.

Since the Green's function G_β is periodic (2.44) and depends only on coordinate differences $G_\beta(x^0 - y^0, \vec{x} - \vec{y})$, it may be represented by Fourier series and integral as

$$G_\beta(x-y) = (1/\beta)\sum_n \exp[i\omega_n(x^4 - y^4)] \int \frac{d^3k}{(2\pi)^3} \exp[i\vec{k}(\vec{x}-\vec{y})]$$

$$\times G_\beta(\omega_n, \vec{k}) \tag{2.45}$$

with $\omega_n = 2\pi n/\beta$.

In compact form one can write this equation as

$$G_\beta(x-y) = \bar{\int_k} \exp[ik(x-y)] G_\beta(k) \tag{2.46}$$

where

$$\bar{\int_k} f(k) = \frac{1}{\beta}\sum_n \int \frac{d^3k}{(2\pi)^3} f(\omega_n, \vec{k}) \tag{2.47}$$

The Green's function satisfies the equation

$$\left(-\partial_x^2 + m^2\right) G_\beta(x-y) = \delta(x-y) \tag{2.48}$$

with periodic conditions (2.44) and δ -function in the form:

$$\delta(x) = \sum_n \int \frac{d^3k}{(2\pi)^3} \exp[i(\omega_n x^4 + \vec{k}\vec{x})] \tag{2.49}$$

From (2.46) and (2.48) it follows that $G_\beta(k)$ is

$$G_\beta(k) = \frac{1}{k^2 + m^2} \tag{2.50}$$

and

$$G_\beta(x) = \bar{\int_k} \frac{\exp(ikx)}{k^2 + m^2} \tag{2.51}$$

2.4 Notation

To work with the standard quantum field theory of signature (+2), return to imaginary time formalism, on the time variable $t = -i\tau$ defined in the interval $[0, -i\beta]$. Therefore the equation (2.44) may be rewritten as

$$G_\beta(x-y)_{|x^0=0} = G_\beta(x-y)_{|x^0=-i\beta} \tag{2.52}$$

It leads to the replacement $\omega_n \to i\omega_n$ and

$$\int_k f(k) = \frac{i}{\beta}\sum_n \int \frac{d^3k}{(2\pi)^3} f(\omega_n, \vec{k}) \tag{2.53}$$

with $\omega_n = 2\pi i n/\beta$.

The equation for the Green's function becomes

$$G_\beta(x) = -\int\limits_k \frac{\exp(ikx)}{k^2 + m^2} \tag{2.54}$$

This expression looks like the expression for the Green's function in a common field theory.

2.5 Partition function for fermions

The previous section discussed a quantization scheme for Bose fields in the functional integral formulation. However, the methods of path integral may be applied in the same way to finite temperature Fermi systems. This section develops these methods.

Consider a free spinor field which is described by the action

$$S = -\int d^4x\bar{\psi}(x)(i\gamma \cdot \partial + m)\psi(x) \tag{2.55}$$

where $\{\gamma_\mu\}$ is a set of Dirac matrices, which are defined by

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu} \tag{2.56}$$

metric $g_{\mu\nu} = diag(-,+,+,+)$and, by notation, $\bar{\psi} = \psi^+\gamma^0$ is hermition conjugate.

The action (2.55) has a global $U(1)$ symmetry and that is associated with conserved current

$$j^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x) \tag{2.57}$$

The total conserved charge is

$$Q = \int d^3xj^0 = \int d^3x\psi^+\psi \tag{2.58}$$

To apply the formalism developed in chapter II.2 treat the field ψ as the basic field, and the momentum congugate to this field

$$\pi = \frac{\partial L}{\partial(\partial\psi/\partial t)} = i\psi^{+} \tag{2.59}$$

Thus consider ψ and ψ^{+} to be independent entities in the Hamiltonian formulation.

The Hamiltonian density is found by the standard procedure as

$$H = \pi\frac{\partial\psi}{\partial t} - L = \psi^{+}\left(i\frac{\partial\psi}{\partial t}\right) - L = \bar{\psi}(-i\gamma\partial + m)\psi \tag{2.60}$$

Introduce the partition function as

$$Z = \mathrm{Tr}\exp\left[-i\beta(H - \mu Q)\right] \tag{2.61}$$

Follow the steps leading up to the equation (2.21). For the partition function the intermediate equation will be:

$$Z_\beta \propto \lim_{N\to\infty}\int\left(id\psi^{+}\frac{d\psi}{2\pi}\right)\times$$

$$\exp\left\{-i\Delta t\sum_{j=1}^{N}\int d^3x\left[H(\psi^{+},\psi) - \psi_j^{+}(\psi_{j+1} - \psi_j)\Delta t\right]\right\} \tag{2.62}$$

Taking into account the limit of this equation, find

$$Z_\beta = N^{'}\int id\psi^{+}\int d\psi$$

$$\exp\left\{ i\int_0^{t_f} dt \int d^3x \left(i\psi^+ \frac{\partial\psi}{\partial t} - H(\psi^+(x,t), \psi(x,t)) \right) \right\} \tag{2.63}$$

For the time variable τ, get

$$Z_\beta = N' \int id\psi^+ \int d\psi$$

$$\exp\left\{ \int_0^{\beta} d\tau \int d^3x \psi^+ \left(\mu - \frac{\partial}{\partial\tau} + i\gamma^0\vec{\gamma}\cdot\vec{\nabla} - m\gamma^0 \right) \psi \right\} \tag{2.64}$$

The quantization of a Fermi system can be obtained as a result of integration over the space of anticommuting functions $\psi(\vec{x},\tau)$ $(\vec{x} \in V, \tau \in [0,\beta])$, which are the elements of an infinite Grassman algebra.

To obtain the correct result it is neccessary to impose on ψ, $\bar{\psi}$ the antiperiodicity conditions in the variable τ:

$$\psi(\vec{x},\beta) = -\psi(\vec{x},0), \quad \bar{\psi}(\vec{x},\beta) = -\bar{\psi}(\vec{x},0) \tag{2.65}$$

The Fourier series for ψ, $\bar{\psi}$ in Fermi case (2.65) are

$$\psi(x,\tau) = (\beta V)^{-1/2} \sum_n \sum_{\vec{k}} \exp(i[\omega\tau + \vec{k}\vec{x}])\psi_n(\vec{k})$$

$$\psi^+(x,\tau) = (\beta V)^{-1/2} \sum_n \sum_{\vec{k}} \exp(-i[\omega\tau + \vec{k}\vec{x}])\psi_n^+(\vec{k}) \tag{2.66}$$

where $\psi_n(\vec{k})$ and $\psi_n^+(\vec{k})$ are the generators of Grassmann algebra.

Inserting (2.66) into (2.64) get

$$Z_\beta = N' \left[\prod_i \prod_n \int i d\psi_n^+ d\psi_n\right] \times$$

$$\exp \sum_n \sum_{\vec{k}} \psi_n^+ \left[(-i\omega_n + \mu) - \gamma^0 \vec{\gamma} \cdot \vec{k} - m\gamma^0\right] \psi_n \tag{2.67}$$

Calculation of the Gaussian functional integral over the fermi fields in (2.67) leads to the following equation

$$Z_\beta = Det[(-i\omega_n + \mu) - \gamma^0 \vec{\gamma} \cdot \vec{k} - m\gamma^0] \tag{2.68}$$

or using useful relation

$$\ln Det D = \mathrm{Tr} \ln D \tag{2.69}$$

find

$$\ln Z_\beta = \mathrm{Tr} \ln[(-i\omega_n + \mu) - \gamma^0 \vec{\gamma} \cdot \vec{k} - m\gamma^0] =$$

$$= 2 \sum_n \sum_{\vec{k}} \ln \left[(\omega_n + i\mu)^2 + \epsilon_{\vec{k}}^2\right] \tag{2.70}$$

Since both positive and negative frequencies are summed over, (2.70) can be written in the form

$$\mathrm{ln} Z_\beta = \sum_n \sum_{\vec{k}} \left\{ \ln \left[(\omega_n - \mu)^2 + \epsilon_{\vec{k}}^2 \right] + \ln \left[(\omega_n + \mu)^2 + \epsilon_{\vec{k}}^2 \right] \right\} \tag{2.71}$$

Summation (2.71) over ω_n leads to the folowing equation

$$\ln Z_\beta =$$

$$2V \sum_{\vec{k}} \left[\beta \epsilon_{\vec{k}} + (1 + \exp\{-\beta(\epsilon_{\vec{k}} - \mu)\}) + (1 + \exp\{-\beta(\epsilon_{\vec{k}} + \mu)\}) \right] \tag{2.72}$$

The contributions for particles (μ) and antiparticles ($-\mu$), and, also, the zero-point energy of vacuum, have now been obtained. [Kapusta 1989]

2.6 Green's function of fermi field

The finite temperature fermionic Green's function (or fermionic propagator) may be introduced in the similar way to scalar field.

Consider that

$$G_\beta^F(x - y) =< T\psi(x), \bar{\psi}(y) >_\beta$$

$$= \mathrm{Tr}[\exp(-\beta H) \mathrm{T}\psi(x), \bar{\psi}(y)] / \ \mathrm{Tr} \ \exp(-\beta H) \tag{2.73}$$

For fermions the analog of (2.39) is in the form

$$G_\beta^F(x - y) = \theta(x - y) G_\beta^{+F}(x - y) + \theta(y - x) G_\beta^F(x - y) \tag{2.74}$$

where

$$G_\beta^{+F}(x-y) =< \psi(x)\bar{\psi}(y) >_\beta, \quad x^4 > y^4$$

and
$$G_\beta^{-F}(x-y) = - < \bar{\psi}(y)\psi(x) >_\beta, \qquad x^4 < y^4 \tag{2.75}$$

The desired boundary conditions now follow from

$$G_\beta^{-F}(x-y) = -\mathrm{Tr}[\exp(-\beta H)\bar{\psi}(y^4,\vec{y})\psi(0,\vec{x})]/\mathrm{Tr}\ \exp(-\beta H)$$

$$= -\mathrm{Tr}[\exp(-\beta H)\exp(\beta H)\psi(0,\vec{x})\exp(-\beta H)\bar{\psi}(y^4,\vec{y})]/\mathrm{Tr}\ \exp(-\beta H)$$

$$= -\mathrm{Tr}[\exp(-\beta H)\psi(\beta,\vec{x})\bar{\psi}(y^4,\vec{y})]/\mathrm{Tr}\exp(-\beta H) = -G_\beta^{+F}(x-y) \tag{2.76}$$

It leads to antiperiodic boundary conditions

$$G_\beta^F(x-y)_{|x^4=0} = -G_\beta^F(x-y)_{|x^4=\beta} \tag{2.77}$$

and this means that the fields ψ, $\bar{\psi}$ are antiperiodic

$$\psi(\vec{x},0) = -\psi(\vec{x},\beta) \tag{2.78}$$

The Fourier series of the fermionic propagator is written as

$$G^F_\beta(x-y) = (1/\beta)\sum_n \exp\ [i\omega_n(x^4-y^4)] \int \frac{d^3k}{2\pi^3} \exp[i\vec{k}(\vec{x}-\vec{y})] G^F_\beta(\omega_n,\vec{k})$$

$$= \bar{\int_k} \exp[ik(x-y)] G^F_\beta(k) \tag{2.79}$$

with $k^\mu = (\omega_n, \vec{k}),\ \ \omega_n = (2\pi/\beta)(n+1/2)$.

The Green's function of the fermionic field satisfies the equation

$$(i\bar{\gamma}\cdot\partial + m) G^F_\beta(x-y) = \delta(x-y) \tag{2.80}$$

and has the following form

$$G^F_\beta(x) = \bar{\int_k} \frac{\bar{\gamma}\cdot k + m}{k^2+m^2} \exp(ikx) \tag{2.81}$$

where $\{\bar{\gamma}_\mu\}$ is the set of Euclidean gamma matrices.

2.7 Notation

As in the case of the scalar field one can work with a quantum model in Minkowski space-time, restoring the time variable $t = -i\tau$ defined in the interval $[0, -i\beta]$. It gives the following antiperiodic conditions for Green's function

$$G^F_\beta(x-y)_{|x^0=0} = -G^F_\beta(x-y)_{|x^0=-i\beta} \tag{2.82}$$

and the rule of integration

$$\int_k f(k) = \frac{i}{\beta}\sum_n \int \frac{d^3k}{(2\pi)^3} f(\omega_n, \vec{k}) \tag{2.83}$$

with $\omega_n = (2\pi i/\beta)(n + 1/2)$. In the Minkowski metric (signature (+2)) the equation for Green's function is written as [Dolan, Jackiw 1974]

$$G^F_\beta(x) = \bar{\int}_k \frac{\gamma \cdot k + m}{k^2 + m^2} \exp(ikx) \tag{2.84}$$

Chapter 3

THERMODYNAMICS OF QUANTUM GASES

AND GREEN'S FUNCTIONS

This section develops a method for calculation of thermodynamic potentials directly from finite temperature Green's functions. For this purpose the Schwinger proper time formalism [Schwinger 1951] has been applied, to write down generating functionals of quantum fields in terms of Green's functions of these fields. The finite temperature generalization of this formalism leads to thermodynamical potentials which are written through finite temperature Green's functions.

3.1 Thermal bosonic fields

In standard form the generating functional Z of the scalar field φ with the Lagrangian (2.26) is written as :

$$Z[J] = \int D\varphi \exp\left\{-(i/2)\int d^4x\varphi(x)K_{xy}\varphi(y) + i\int d^4xJ(x)\varphi(x)\right\} \tag{3.1}$$

The symmetric operator

$$K_{xy} = \left(-\partial_x^2 + m^2\right)\delta(x-y) \tag{3.2}$$

can formally be treated as a symmetric matrix with continuous indices (x, y). It has the following properties:

$$\int d^4yK_{xy}^{1/2}K_{yz}^{-1/2} = \delta(x-z)$$

$$\int d^4yK_{xy}^{1/2}K_{yz}^{1/2} = K_{xz} \tag{3.3}$$

The functional Z gives the transition amplitude from the initial $|0^- >$ and final $|0^+ >$ vacuum in the presence of the source $J(x)$, $(Z =< 0^+|0^- >_J)$ which is producing particles. The Green's function may be treated as the solution of the equation

$$\int d^yK_{xy}G(y,x^{'}) = \delta(x-x^{'}) \tag{3.4}$$

with operator K_{xy}. Now rewrite (3.1) in a convenient form changing the integration variable from φ to

$$\varphi^{'}(x) = \int d^4 y K_{xy} \varphi(y) \tag{3.5}$$

Then the quadratic form of (3.1) may be recast as

$$- (1/2) \int d^4 x \left[\varphi^{'}(x) - \int d^4 y K_{yx}^{-1/2} J(y) \right]$$

$$- (1/2) \int d^4 x d^4 y \varphi(y) G(x, y) \varphi(y) \tag{3.6}$$

Substituting (3.6) into (3.1) and performing an integration of the Gaussian type integral, get

$$Z[J] \propto (det K^{1/2})^{-1} \exp \left[-(i/2) \int d^4 x d^y \varphi(y) G(x, y) \varphi(y) \right] \tag{3.7}$$

where the Jacobian arises from the change of variable (3.5).

The functional determinant in (3.7) may be written with Green's function $G(x, y)$:

$$(Det K^{1/2})^{-1} = (Det G(x, y))^{1/2} = \exp \left[(1/2) \mathrm{Tr} \ln G(x, y) \right] \tag{3.8}$$

The Green's function is found from functional differentiation of Z with respect to source J

$$(i)^2 \left(\frac{\delta \ln Z}{\delta J(x) \delta J(y)} \right)_{J=0} =< 0|\mathrm{T}\varphi(x)\varphi(y)| >= G(x,y) \tag{3.9}$$

Now pay attention to the functional determinant in (3.7) because of its important role in the applications to statistical mechanics.

One can introduce the heat kernel for operator K as the solution of the partial differential equation

$$i\frac{\partial}{\partial s}\Im(x,x^{'};is) = \int d^4 z K_{xz} \Im(z,x^{'};is) \tag{3.10}$$

with the boundary conditions $\Im(x,x^{'};0) = \delta(x-x^{'})$

The Green's function G may be written with $\Im$ in the form

$$G(x,x^{'}) = \int\limits_0^\infty ids \Im(z,x^{'};is) \tag{3.11}$$

and the logarithm of functional determinant of the operator K as

$$\ln DetK = \int\limits_0^\infty ids(is)^{-1} \mathrm{tr} \Im(z,x^{'};is) \tag{3.12}$$

To get the equation for the heat kernel $\Im$, find proper time representation of Green's function.

Using the useful equation

$$(k^2+m^2)^{-1} = i\int\limits_0^\infty ds \exp\{-is(k^2+m^2)\} \tag{3.13}$$

write the equation for Green's function in the form

$$G(x, x^{'}) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + m^2} \exp(iky)$$

$$= i \int\limits_0^\infty ds \int \frac{d^4k}{(2\pi)^4} \exp\{iky - is(k^2 + m^2)\} \qquad (3.14)$$

where $y = x - x^{'}$.

After the integration with respect to momentum, we get the Schwinger representation of the Green's function

$$G(x, x^{'}) = \frac{i}{(4\pi)^2} \int\limits_0^\infty ids (is)^2 \exp(-ism^2 - \sigma/2is) \qquad (3.15)$$

The variable σ equals half the square of the distance between x and $x^{'}$

$$\sigma = (x - x^{'})^2/2$$

The equation for the heat kernel directly follows from the comparision (3.11) and (3.15):

$$\Im(x, x^{'}; is) = \frac{i}{(4\pi is)^2} \exp(-ism^2 - \sigma/2is) \qquad (3.16)$$

For future calculations it is better to work with the generating functional of connected Green's functions $W[J]$ which is connected with generating functional Z by means of equation

$$Z[J] = \exp\{iW[J]\} \tag{3.17}$$

From (3.12) and (3.17) we find

$$W[0] = (i/2)\ln DetK$$

$$= -(i/2)\int\limits_0^\infty ids(is)^{-1}\mathrm{tr}\Im(x,x^{'};is) \tag{3.18}$$

Inserting (3.16) into (3.18) and taking into account (3.15) we get an important equation

$$W[0] = -(i/2)\int d^4x \int\limits_{m^2}^\infty dm^2 \mathrm{tr}G(x,x^{'}) \tag{3.19}$$

Now find finite temperature functional W_β inserting Green's function at finite temperature

$$W_\beta = -(i/2)\int\limits_\beta d^4x \int\limits_{m^2}^\infty dm^2 \mathrm{tr}G_\beta(x,x^{'}) \tag{3.20}$$

Now

$$W_\beta = -(\beta/2)\int\limits_V d^3x \int\limits_{m^2}^\infty dm^2 \int\limits_k (k^2+m^2)^{-1}$$

$$= (\beta/2)V \int_k \ln(k^2 + m^2) \tag{3.21}$$

The Helmholtz free energy may be treated as effective potential of the finite part of W_β, therefore

$$F(\beta, V) = -iW_\beta. \tag{3.22}$$

and

$$F(\beta, V) = -(i\beta/2)V \int_k \ln(k^2 + m^2)$$

$$= (1/2)V \int \frac{d^3k}{(2\pi)^3} \sum_n \ln(\omega_n^2 + \epsilon_k^2) \tag{3.23}$$

After making the summation over the frequencies ω, this equation will coincide with (2.37) (if the infinite, temperature independent term is dropped).

So one can conclude that the density of Hemholtz free energy may be written as [Kulikov & Pronin 1987]

$$f(\beta) = (i/2) \int_{m^2}^{\infty} dm^2 \mathrm{tr} G(\beta, x - x'), \tag{3.24}$$

where $G(\beta, x - x')$ is temperature contribution in $G_\beta(x, x')$.

3.2 Bosonic finite temperature Green's function in the Schwinger representation

Start from the finite temperature Green's function (2.51) in the limit of coincidence $x = x^{'}$:

$$\lim_{x \to x^{'}} G_{\beta}(x, x^{'}) = \bar{\int_k} \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + m^2}$$

$$= -\int_0^{\infty} ds (1/\beta) \sum_n \int \frac{d^3k}{(2\pi)^3} \exp\{-is(-\omega_n^2 + \vec{k}^2 + m^2)\} \tag{3.25}$$

After the integration with respect to momentum find

$$\lim_{x \to x^{'}} G_{\beta}(x, x^{'}) = \int_0^{\infty} (4\pi is)^{3/2} \exp(-ism^2)(1/\beta) \sum_n \exp(-4\pi^2 n^2/\beta^2), \tag{3.26}$$

Rewrite the sum in (3.26) with the help of the equation [Ditrich 1978]

$$\sum_n \exp[-\alpha(n - z)^2] = \sum_n (\pi/\alpha)^{1/2} \exp(-\pi^2 n^2/\alpha - 2\pi izn) \tag{3.27}$$

In bosonic case ($z = 0$) (3.26) has the form:

$$G_{\beta}(x, x) = \frac{i}{(4\pi)^2} \sum_n \int_0^{\infty} \frac{ids}{(is)^2} \exp(-ism^2 - n^2\beta^2/4is) \tag{3.28}$$

Selecting the temperature independent ($n = 0$) part of the equation (3.28) we find the finite temperature contribution in Green's function of scalar field:

$$G_{\beta}(x, x) = G(x, x) + G(\beta, x - x)$$

$$= \frac{i}{(4\pi)^2} \int_0^\infty \frac{ids}{(is)^2} \exp(-ism^2)$$

$$+ \frac{i}{(4\pi)^2} 2 \sum_{n=1}^{\infty} \int_0^\infty \frac{ids}{(is)^2} \exp(-ism^2 - n^2\beta^2/4is) \tag{3.29}$$

Inserting $G(\beta, x - x)$ of the equation (3.29) into (3.24) we find the equation for free energy in the form

$$f(\beta) = -(1/(4\pi)^2) \int_{m^2}^{\infty} dm^2 \sum_{n=1}^{\infty} \int_0^\infty \frac{ids}{(is)^2} \exp(-ism^2 - n^2\beta^2/4is) \tag{3.30}$$

The integral over (s) occuring in (3.30) can be found with

$$\int_0^\infty dx x^{\nu-1} \exp\left(-\alpha \frac{1}{x} - \gamma x\right) = 2\,(\alpha/\gamma)^{\nu/2}\, K_\nu(2\sqrt{\alpha\gamma}), \tag{3.31}$$

where K_ν is a modified Bessel function. Then

$$\int_0^\infty ids(is)^{j-3} \exp(-ism^2 - \beta^2 n^2/4is) = 2\,(\beta n/2m)^{j-2}\, K_{j-2}(\beta mn), \tag{3.32}$$

and the equation for f will be

$$f(\beta) = -(m^2/2\pi^2\beta^2) \sum_{n=1}^{\infty} (1/n^2) K_2(\beta mn) \tag{3.33}$$

The integral representation of the sum of the modified Bessel function (XXII.32) allows the standard equation for the Helmholtz free energy to be written as

$$f(\beta) = (1/\beta) \int \frac{d^3k}{(2\pi)^3} \ln(1 - \exp(-\beta\epsilon_k)) \quad (3.34)$$

directly from (3.33).

3.3 Thermal fermionic fields

Let us now develop the same formalism for fermionic fields.

Add to the fermionic Lagrangian the terms with anticommutative sources $\eta(x)$ and $\bar{\eta}(x)$

$$L \to L = L_0 + \eta(x)\bar{\psi}(x) + \psi(x)\bar{\eta}(x) \quad (3.35)$$

and write the generating functional

$$Z[\eta, \bar{\eta}] = \int D\psi D\bar{\psi} \exp\left\{i \int d^4x \left[L_0 + \eta(x)\bar{\psi}(x) + \psi(x)\bar{\eta}(x)\right]\right\}, \quad (3.36)$$

where the functions ψ and $\bar{\psi}$ are considered as Grassman variables.

After integration in respect with these Grassmann variables obtain

$$Z[\eta, \bar{\eta}] \propto (DetK) \exp\left[-i \int d^4x \int d^4y \bar{\eta}(x) K_{xy} \eta(y)\right] \quad (3.37)$$

where

$$K_{xy} = (i\gamma \cdot \partial_x + m)\delta(x - y) \quad (3.38)$$

and bi-spinor G_F satisfies the equation which is equivalent to (3.4).

In operator form [Schwinger 1951] it is written

$$KG_F = \hat{1} \tag{3.39}$$

At zero sources the generating functional $W[0]$ is

$$W[0] = -i \ln Z[0] = -i \ln Det(K) \tag{3.40}$$

On the other hand it is

$$W[0] = -i \ln Det(K)$$

$$= (i/2) \int_0^\infty ids (is)^{-1} \mathrm{tr} \hat{\Im}(x, x^{'}; is) \tag{3.41}$$

where the kernel $\hat{\Im}(x, x^{'}; is)$ is the solution of the equation

$$i \frac{\partial}{\partial s} \hat{\Im}(x, x^{'}; is) = (K \hat{\Im})(x, x^{'}; is), \tag{3.42}$$

The equation for the bi-spinor Green's function is written as

$$G_F(x, x^{'}) = \int \frac{d^4k}{(2\pi)^4} \frac{\hat{1}}{k^2 + m^2} \exp(iky)$$

$$= i\int_0^\infty ds \int \frac{d^4k}{(2\pi)^4} \exp\{iky - is(k^2+m^2)\} \tag{3.43}$$

and after integration over the momentum, we get

$$G_F(x,x^{'}) = -\frac{\hat{1}}{(4\pi)^2}\int_0^\infty ds(is)^2 \exp(-ism^2 - \sigma/2is), \tag{3.44}$$

As follows from (3.11) the heart kernel has the form

$$\hat{\Im}(x,x^{'};is) = -i\frac{\hat{1}}{(4\pi)^2}(is)^2 \exp(-ism^2 - \sigma/2is), \tag{3.45}$$

Inserting (3.45) into (3.41) and using (3.44) we find

$$W[0] = (i/2)\int d^4x \int_{m^2}^\infty dm^2 \mathrm{tr} G_F(x,x^{'}) \tag{3.46}$$

For non-interacting fermionic field this equation is divergent and does not give any useful information but, in the finite temperature case we can get some interesting results connected with temperature properties of quantum Fermi gas.

3.4 Fermionic finite temperature Green's function in Schwinger representation

In the limit of coincidence ($x \to x^{'}$) the finite temperature fermionic bi-spinor G_F is written as:

$$\lim_{x \to x'} G_F(x, x') = \int \frac{d^4k}{(2\pi)^4} \frac{\hat{1}}{k^2 + m^2}$$

$$= - \int_0^\infty ds (\hat{1}/\beta) \sum_n \int \frac{d^3k}{(2\pi)^3} \exp\{-is(-\omega_n^2 + \vec{k}^2 + m^2)\} \tag{3.47}$$

where $\omega_n = (2i\pi n/\beta)(n + 1/2)$.

After the integration in respect with momentum we get

$$\lim_{x \to x'} G_F(x, x') =$$

$$\int_0^\infty (4\pi is)^{3/2} \exp(-ism^2)(\hat{1}/\beta) \sum_n \exp[-4\pi^2(n + 1/2)^2/\beta^2] \tag{3.48}$$

Using the equation (3.27) with $z = -1/2$, we find

$$\sum_n \exp[-(2\pi/\beta)^2(n + 1/2)^2] = \sum_n (-1)^n [\beta/(4\pi is)^{1/2}] \exp(-\pi^2 n^2/4is), \tag{3.49}$$

Then the equation (3.48) will be

$$\lim_{x \to x'} G(x, x') = i \frac{\hat{1}}{(4\pi)^2} \sum_n (-1)^n \int_0^\infty \frac{ids}{(is)^2} \exp(-ism^2 - n^2\beta^2/4is). \tag{3.50}$$

Now we can select the temperature independent part of the equation (3.50):

$$G_F^\beta(x, x) = G_F(x, x) + G_F(\beta, x - x)$$

$$= i\frac{\hat{1}}{(4\pi)^2}\int_0^\infty \frac{ids}{(is)^2}\exp(-ism^2)$$

$$+ i\frac{\hat{1}}{(4\pi)^2}2\sum_{n=1}^\infty(-1)^n\int_0^\infty \frac{ids}{(is)^2}\exp(-ism^2 - n^2\beta^2/4is) \tag{3.51}$$

Inserting (3.51) into the equation [Kulikov & Pronin 1987]:

$$f = (-i/2)\int_{m^2}^\infty dm^2 \mathrm{tr} G_F(\beta, x - x')) \tag{3.52}$$

we find an expression for the free energy of Fermi gas in the form

$$f(\beta) = -(1/(4\pi)^2)\mathrm{tr}(\hat{1})\int_{m^2}^\infty dm^2\sum_{n=1}^\infty(-1)^n\int_0^\infty \frac{ids}{(is)^2}\exp(-ism^2 - n^2\beta^2/4is) \tag{3.53}$$

The integral over (s) occuring in (3.53) can be written as

$$\int_0^\infty ids(is)^{j-3}\exp(-ism^2 - \beta^2 n^2/4is) = 2\left(\beta n/2m\right)^{j-2} K_{j-2}(\beta mn), \tag{3.54}$$

and the equation for density of free energy of fermionic field f will be

$$f(\beta) = \mathrm{tr}(m^2/2\pi^2\beta^2)(\hat{1})\sum_{n=1}^\infty \frac{(-1)^n}{n^2}K_2(\beta mn) \tag{3.55}$$

The standard equation for Helmholtz free energy follows from (XXII.14)

$$f(\beta) = -(4/\beta)\int \frac{d^3k}{(2\pi)^3}\ln(1 + \exp(-\beta\epsilon_k)) \tag{3.56}$$

Now we can describe the thermal behavior of massless vector fields in the Schwinger proper time formalism to prepare the mathematical foundation for further computations in curved space-time.

Chapter 4

FINITE TEMPERATURE GAUGE FIELDS

In the present day progress in quantum field theory is to a great extent due to the development of gauge fields [Yang & Mills 1954, Utiyama 1956, 't Hooft 1971]. These fields open up new possibilities for the description of interactions of the elementary particles. Gauge fields are involved in most modern models of strong, weak and electromagnetic interactions. They are extremely attractive for the unification of all interactions into a single universal interaction. On the other hand the functional formulation of the gauge fields helps us to get statistical and thermodynamical results connected with finite temperature properties of the particles which are described by these fields. In this connection we will develop the finite temperature formalism for gauge fields

4.1 Gauge theories: Pure Yang-Mills theory

The basic idea of gauge field relies on the local gauge invariance principle of the quantum field theory [Yang & Mills 1954, Utiyama 1956]. Let us consider a gauge tansformation parametrized by the functions $\omega^a(x)$

$$U(x) = \exp[-i\omega^a(x)\tau^a] \tag{4.1}$$

where the generators of the Lee algebra obey the equation

$$[\tau^a, \tau^b] = if^{abc}\tau^c \tag{4.2}$$

Numbers f^{abc} are the structure constants of the group.

Let the vector (gauge) fields $A_\mu(x) = A_\mu(x)^a\tau^a$ transform according to

$$A_\mu(x) \to A_\mu^\omega(x) = UA_\mu(x)U^{-1} + (i/g)U\partial_\mu U^{-1} \tag{4.3}$$

For infinitesimal transformations we have

$$(A^\omega)_\mu^a = A_\mu^a(x) + f^{abc}A_\mu^b(x)\omega^c(x) + (1/g)\partial_\mu\omega^a(x)$$

$$= A_\mu^a(x) + \delta A_\mu^a(x) \tag{4.4}$$

The transformations (4.1) preserve the Maxwell type Lagrangian

$$L = -(1/2)\mathrm{Tr}(F_{\mu\nu}(x))^2 = -(1/4)F^a_{\mu\nu}F^{a\mu\nu} \tag{4.5}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$$

$$= (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu)\tau^a \tag{4.6}$$

is the strength of the vector field A_μ.

The path integral over the field A_μ with the Lagrangian (4.5) in the form

$$\int DA_\mu \exp(iS[A_\mu, \partial_\nu A_\mu]) \tag{4.7}$$

is undefined because of the gauge degrees of freedom. Namely, the gauge transformations create the "orbit" of the field A^ω_μ in the functional field space and the functional integration (4.7) over this space overcounts the degrees of freedom of the theory. To improve the situation we have to eliminate all unphysical degrees of freedom which origin from local gauge invariance. So, we have to "slice" the orbit once so that we do not have this infinite overcounting. This is the origin of the gauge fixing problem. To solve this problem one can select the surface in this functional space. The surface is good if it intersects the orbit of any given field under gauge transformation once and only once. The equation of this surface may be written as

$$F(A_\mu) = 0 \tag{4.8}$$

To eliminate the unphysical degrees of freedom we may insert the factor $\delta[F(A_\mu)]$ in the functional integral.

We can do it inserting

$$1 = \Delta_{FP}(A_\mu) \int D\omega \delta[F(A_\mu^\omega)] \tag{4.9}$$

into (4.7). This expression can not change the measure of integration in (4.7) so we will have

$$\int DA_\mu \exp\left\{i \int d^4xL\right\}$$

$$= \int DA_\mu \left(\Delta_{FP}(A_\mu) \int D\omega \delta[F(A_\mu^\omega)]\right) \exp\left\{i \int d^4xL\right\}, \tag{4.10}$$

where δ_{FP} is Faddeev-Popov determinant, and $D\omega = \prod_a \prod_x d\omega^a(x)$ is the invariant group measure [Faddeev & Slavnov 1980].

One can show that it is gauge invariant [Ryder 1985]

$$\Delta_{FP}(A_\mu) = \Delta_{FP}(A_\mu^\omega) \tag{4.11}$$

therefore the equation (4.10) is

$$\int DA_\mu \int D\omega \Delta_{FP}(A_\mu^\omega)\delta[F(A_\mu^\omega)] \exp\left\{i \int d^4x L\right\}. \tag{4.12}$$

Making the gauge transformation from A_μ^ω to A_μ and taking into account that action is gauge invariant too, we get, dropping multiplicative divergent factor $\int d\omega$, the expression for generating functional of the vector field A_μ

$$Z[J] = \int DA_\mu \Delta_{FP}(A_\mu)\delta[F(A_\mu)] \exp\left\{i \int d^4x L + i(J, A)\right\} \tag{4.13}$$

Our task now is to calculate an explicit result of the Faddeev-Popov determinant $\Delta_{FP}(A_\mu)$.

To do this we denote the numerical value of the function $F(A_\mu)$ at x as

$$F(A_\mu) = F^a(x) \tag{4.14}$$

Then one can write the equation (4.9) in the form

$$\Delta_{FP}^{-1}(A) = \prod_x \prod_a \int d\omega^a(x)\ \delta(F^a(x))$$

$$= \prod_x \prod_a \int dF^a(x)\ \delta(F^a(x)) \frac{\partial(\omega_1(x)...\omega_N(x))}{\partial(F_1(x)...F_N(x))}$$

$$= \prod_x \left\| \frac{\partial \omega^a(x)}{\partial F^b(x)} \right\|_{F=0} = Det\left(\frac{\delta\omega}{\delta F}\right)_{|F=0} \tag{4.15}$$

The Faddeev-Popov determinant

$$\Delta_{FP}(A) = Det\left(\frac{\delta F}{\delta\omega}\right)_{|F=0} \tag{4.16}$$

is the functional determinant of the continuous matrix $\parallel \delta F^a(x)/\delta\omega^b(y) \parallel$ with rows labelled by (a, x) and columns by (b, y).

The generating functional (4.13) will have the form

$$Z[J] = \int DA_\mu(x) Det\left(\delta F^a/\delta\omega^b\right) \delta[F^a(A_\mu)] \exp\left\{iS + i(J, A)\right\} \tag{4.17}$$

Now we can make the last step to rewrite our expression in a form convenient for practical calculations introducing Faddeev-Popov (ghost) fields.

4.2 Ghost fields

For the calculation of generating functional (4.17) we must select a gauge constraint and compute the FP-determinant (4.16) in this gauge.

Let us select the Lorentz gauge constrain

$$F^a(A_\mu) = \partial^\mu A^a_\mu(x) = 0 \tag{4.18}$$

We can find $Det(\delta F^a/\delta\omega^b)$ from the Teylor expansion of the F^a with gauge transformation ω^b:

$$F^a \to (F^\omega)^a = F^a + Det(\delta F^a/\delta\omega^b)\omega^b + ... \tag{4.19}$$

For our gauge conditions (4.18) it is

$$\partial^\mu A^a_\mu(x) \xrightarrow{\omega} \partial^\mu A^a_\mu(x) + f^{abc}\partial(A^b_\mu(x)\omega^c(x)) + (1/g)\partial^2\omega^a(x)$$

$$= \partial^\mu A^a_\mu(x) + ((1/g)\partial^2\delta^{ac} + A^{ac}_\mu(x)\partial^\mu)\omega^c(x)$$

$$= \partial^\mu A^a_\mu(x) + \int dy \sum_c \Big[(1/g)\partial^2\delta^{ac}\delta(x-y) + A^{ac}_\mu(x)\partial^\mu\delta(x-y)\Big]\,\omega^c(y) \tag{4.20}$$

where $A^{ac}_\mu = f^{abc}A^b_\mu(x)$.

Then we can find, that

$$Det\,[\delta F/\delta\omega] = Det\;\| < x, a|\delta F/\delta\omega|y, c > \|$$

$$= Det\Big[((1/g)\partial^2\delta^{ac} + A^{ac}_\mu(x)\partial^\mu)\delta(x-y)\Big] \tag{4.21}$$

One can rewrite Faddeev-Popov determinant (4.21) in terms of a path integral over anti-commuting fields $(\bar{c}, c)$[1]. They are called Faddeev-Popov ghosts. the fields $\bar{c}$, c are independent anti-commuting scalar fields which obey Fermi statistics.

[1]The functional determinant of anti-commuting fields $\bar{c}$, c is

$$\int D\bar{c}Dc\exp\{i\bar{c}^a M^{ab} c^b\} = Det(M)$$

Then the determinant (4.21) may be written in the form

$$Det\,[\delta F/\delta\omega] = \int Dc D\bar{c} \exp\left[i\int d^4x L_{ghost}(x)\right] \tag{4.22}$$

Lagrangian $L_{ghost}(x)$ is

$$L_{ghost}(x) = \bar{c}^a(x)(\partial^2\delta^{ab} + gA^{ab}_\mu(x)\partial^\mu)c^b(x) \tag{4.23}$$

The first term in the Lagrangian (4.23) is the pure ghost part

$$L^{(0)}_{ghost}(x) = \bar{c}(x)(\partial^2\delta^{ac})c(x) \tag{4.24}$$

and the ghost propagator G_0 directly follows from (4.24). The equation for this propagator is

$$\partial^2_x G_0(x-y) = \delta(x-y), \tag{4.25}$$

In the momentum space $G_0(k)$ is written

$$G_0(k) = 1/(k^2 + i\epsilon).$$

The second term in (4.23) describes interaction ghosts and the vector field

$$igf^{abc}\partial_\mu \tag{4.26}$$

For better understanding of the ghost contributions let us study an effective action of the ghost fields.

4.3 Effective action

The functional determinant (4.21) may be rewritten as

$$Det(\delta F/\delta\omega) = Det(\partial^2\delta^{ac} + gA_\mu^{ac}\partial_\mu)$$

$$= Det((1/g)\partial^2)Det\left(1 + g(1/\partial^2)A_\mu^{ac}\partial_\mu\right) \tag{4.27}$$

Ignoring the first determinant[2]

$$Det(\partial^2) = Det(G_0^{-1}), \tag{4.28}$$

we can write an effective action in the form

$$S_{eff} = S - iTr\ln(1 + L), \tag{4.29}$$

where the element L is written as

$$< x, a|L|y, c >= g < x, b| \left[(1/\partial^2)A_\mu^{ac}(x)\partial_x^\mu\delta(x - y)\right] |y, c >$$

$$= g\int dzG_0(x - z)A_\mu^{ac}(x)\partial_z^\mu\delta(z - y). \tag{4.30}$$

[2]This term is important in the finite temperature limit.

Now we can expand the determinant with

$$Det(1+L) = \exp\{\mathrm{Tr}\ln(1+L)\} =$$

$$= \exp\left(\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\mathrm{Tr}L^n\right) \tag{4.31}$$

Write the contributions $\mathrm{Tr}(L^n)$ in (4.31).

The first expression of the order g is

$$\ln L = \mathrm{tr}\int dx \int dz \left\{G_0(x-z)gA_\mu^{ac}(z)\partial_z\delta(z-x)\right\} \tag{4.32}$$

because $\mathrm{tr}(A_\mu^{ac}(x)) = \mathrm{tr}(f^{abc}A_\mu^b(x)) = 0$.

The second expression of the order g^2 will be

$$\ln(L^2) = g^2\mathrm{tr}\int dx \int dy \int dz \int dt G_0(x-z)\times$$

$$\times A_\mu^{ac}(z)\partial_z\delta(z-x)G_0(y-t)A_\nu^{ca}(t)\partial_t\delta(t-x)$$

$$= \int dz \int dt\,\{gA_\nu^{ca}(t)\partial_t G_0(t-z)\}\times$$

$$\times\left\{gA_\mu^{ac}(z)\partial_z G_0(t-z)\right\} \tag{4.33}$$

As we can see from (4.33) and (4.31), for $n > 2$ the trace of (4.29) equals the integral with integrand in the form of the cycle multiplication of the terms

$$\left\{gA_\mu^{ac}(z)\partial_z G_0(t-z)\right\} \tag{4.34}$$

namely, we can write

$$\mathrm{Tr}(L^n) = \int dz_1 ... \int dz_n \left\{gA_\mu^{ac_1}(z_1)\partial_{z_1} G_0(z_1 - z_2)\right\} \times ...$$

$$\times ... \left\{gA_\mu^{c_n a}(z_n)\partial_{z_n} G_0(z_{n-1} - z_n)\right\} \tag{4.35}$$

These terms give non-trivial corrections due to ghosts to the effective action $S_{eff}[A]$

Therefore the determinant (4.31) may be treated as contributions of the closed loops with internal ghost lines, and external lines of which correspond to vector fields.

4.4 Propagator of vector field

In the previous section we found out how to compute the Faddeev-Popov determinant in the generating functional (4.17). Now we must treat the second important term the delta function arising from the gauge constraint. For this we will rewrite the delta function as [Vladimirov 1976]

$$\delta(\partial^\mu A_\mu^a) = \prod_x \delta(\partial^\mu A_\mu^a(x))$$

$$= \lim_{\alpha\to 0} \prod_x (-2i\pi\alpha)^{-1/2} \exp\left[-\frac{i}{2\alpha}(\partial^\mu A^a_\mu(x))^2\right]$$

$$\sim \lim_{\alpha\to 0} \exp\left\{-\frac{i}{2\alpha}\int d^4x \left[\partial^\mu A^a_\mu(x)\right]^2\right\} \tag{4.36}$$

The argument of the exponent (4.36) may be combined with the quadratic part of the vector Lagrangian (4.5):

$$i\int d^4x \left[-\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 - \frac{1}{2\alpha}\left(\partial^\mu A^a_\mu(x)\right)^2\right] \tag{4.37}$$

The second term of the expression (4.37)fixes the gauge.

As a result we have the quadratic part over the vector fields in the form

$$-\frac{i}{2}\int d^4x A^a_\mu(x)\left[(\eta^{\mu\nu}\partial^2 + \left(1-\frac{1}{\alpha}\right)\partial_\mu\partial_\nu)\right]A^a_\nu(x) \tag{4.38}$$

The propagator of the vector field may be found from the equation

$$\left[(\eta^{\mu\nu}\partial^2 + \left(1-\frac{1}{\alpha}\right)\partial_\mu\partial_\nu)\right]G_{\nu\lambda}(x-y) = -\delta_{\nu\lambda}\delta(x-y). \tag{4.39}$$

The solution of this equation is

$$G_{\mu\nu}(x) = \int \frac{d^4k}{(2\pi^4)}\left[\eta_{\mu\nu} + (1-\alpha)\frac{k_\mu k_\nu}{k^2}\right]\frac{\exp(ikx)}{k^2+i\epsilon} \tag{4.40}$$

The finite temperature propagator of the vector field in the Feynman gauge ($\alpha = 1$)

has the simple form

$$G_{\mu\nu}(x) = \int\limits_k \frac{\eta_{\mu\nu}}{k^2 + i\epsilon} \exp(ikx) \tag{4.41}$$

where the definition of the integral $\int\limits_k$ is the same as in (2.54)

4.5 Partition function for Gauge fields

Let us study thermal gauge fields in the axial gauge ($A_3 = 0$).

The standard expression for the partition function is

$$Z = \mathrm{Tr}\exp[-\beta H]$$

where $H = H(\pi_i^a, A_i^a)$ is the Hamiltonian of the Yang-Mills field.

According to section II (2.15) we can write

$$Z = (\mathrm{Tr}\exp[-\beta H])_{axial} \tag{4.42}$$

$$= N \int \prod_a d\pi_1^a d\pi_2^a \int\limits_{periodic} dA_1^a dA_2^a \times$$

$$\times \exp\left\{ \int\limits_0^\beta d\tau \int d^3x [i\pi_i \dot{A}_i^a - H(\pi_i^a, A_i^a)] \right\} \tag{4.43}$$

The Hamiltonian in this gauge has only two degrees of freedom for each vector field: $A_{1,2}^a$. The conjugate momenta for these fields are $\pi_{1,2}^a$.

After Gaussian integration over the momenta, we get

$$Z = N \int\limits_{periodic} DA^a \prod_a \delta[A_3^a] \exp \left\{ \int\limits_0^\beta d\tau \int d^3x L(x) \right\} \tag{4.44}$$

The result is similar to the expression for the Faddeev-Popov ansatz in the axial gauge. Calculations of (4.44) give the correct result for the partition function of a photon gas. The same result will also be obtained for Coulomb gauge ($\vec{\nabla}\vec{A} = 0$).

For the Feynman gauge the simple calculation of the type (4.43) leads to an incorrect result. But we may correct the result taking into consideration the ghost contribution and rewrite the partition function (4.43) in the form of (4.13).

$$Z = N \int\limits_{periodic} DA^a Det(\delta F^a/\delta\omega^b)\delta[F^a] \exp \left\{ \int\limits_0^\beta d\tau \int d^3x L(x) \right\} \tag{4.45}$$

Since under the gauge transformations $\delta A_\mu^a = -\partial_\mu \omega^a$, we have

$$Det(\delta(\partial_\mu A^{a\mu})/\delta\omega^b) = Det(\partial^2) \tag{4.46}$$

that coincides with (2.28) at order $\sim 0(g)$. For the Feynman gauge ($\alpha = 1$)

$$\int\limits_{periodic} DA^a \exp \left\{ \int\limits_0^\beta d\tau(-1/2) \int d^3x \left[A_\mu^a(x)\partial^2 A^{a\mu(x)} \right] \right\} = Det(G_{\mu\nu})^{1/2} \tag{4.47}$$

At finite temperature $Det(G_{\mu\nu})$ may be written in the form

$$Det(G_{\mu\nu}) = \exp(-1/2)\mathrm{Tr}\ln(G_{\mu\nu})$$

$$= \exp(-2) \sum_n \int \frac{d^3k}{(2\pi)^3} \ln(\omega_n^2 + \vec{k}^2) \tag{4.48}$$

and the temperature contribution of $Det(\partial^2)$ in the form

$$Det(\partial^2) = \exp \mathrm{Tr} \ln(\partial^2) = \exp \mathrm{Tr} \ln(G)$$

$$= \exp \sum_n \int \frac{d^3k}{(2\pi)^3} \ln(\omega_n^2 + \vec{k}^2) \tag{4.49}$$

where G is just the propagator of the scalar field.

The logarithm of the finite temperature generating functional $Z[0]$, according to previous results, is

$$\ln Z[0] = \ln Det(G) - (1/2) \ln Det(G_{\mu\nu}), \tag{4.50}$$

or

$$\ln Z[0] = -\sum_n \int \frac{d^3k}{(2\pi)^3} \ln(\omega_n^2 + \vec{k}^2). \tag{4.51}$$

After summation the free energy is found in the form

$$f(\beta) = -\frac{1}{\beta} \ln Z_\beta = \left(\frac{2}{\beta}\right) \int \frac{d^3k}{(2\pi)^3} \ln(1 - \exp[-\beta\epsilon]), \tag{4.52}$$

where $\epsilon^2 = (\vec{k})^2$.

So, we got the correct answer [Bernard 1974] by introducing the finite temperature ghost contribution. The index (2) in the equation (4.52) reflects the two degrees of freedom of the massless vector field. The generalization to Schwinger proper time formalism is the same as in the previous section.

Chapter 5

QUANTUM FIELDS

IN CURVED SPACE-TIME

In this chapter we introduce the basic formalism of quantum fields in curved space-time which was studied in works of DeWitt, Fulling, Bunch et al. [DeWitt 1975], [Fulling 1989], [Parker 1979], [Birrell & Davies 1982].

5.1 Lorentz group and quantum fields

Field theory in Minkowski space-time has been well studied from the group point of view by many authors [Sterman 1993],[Ramond 1983]. Here we will consider the connection of fields of different spin with the Lorentz group.

In flat space the spin of the field is classified according to the field's properties under infinitesimal Lorentz transformations

$$\bar{x}^\alpha = \Lambda^\alpha_\beta x^\beta = (\delta^\alpha_\beta + \omega^\alpha_\beta)x^\beta \tag{5.1}$$

with

$$\omega_{\alpha\beta} = \omega_{\beta\alpha}$$

which preserve the length of the coordinate vector x^2.

Under Lorentz transformations the general multicomponent field

$$F^{\alpha\beta\ldots\lambda}$$

transforms according to

$$F^{\alpha\beta\ldots\lambda} \xrightarrow{\Lambda} [D(\Lambda)]^{\alpha\beta\ldots\lambda}_{\alpha'\beta'\ldots\lambda'} F^{\alpha'\beta'\ldots\lambda'} \tag{5.2}$$

where

$$[D(\Lambda)] = 1 + (1/2)\omega^{\alpha\beta}\Sigma_{\alpha\beta}. \tag{5.3}$$

In order for the Lorentz transformations to form a group, the antisymmetric $\Sigma_{\alpha\beta}$ group generators are constrained to satisfy

$$[\Sigma_{\alpha\beta}, \Sigma_{\gamma\delta}] = \eta_{\gamma\beta}\Sigma_{\alpha\delta} - \eta_{\alpha\gamma}\Sigma_{\beta\delta} + \eta_{\delta\beta}\Sigma_{\gamma\alpha} - \eta_{\delta\alpha}\Sigma_{\gamma\beta} \tag{5.4}$$

We may write down $\Sigma_{\alpha\beta}$ for different types of fields:

1) It is easy to see that the scalar field has the following form of Σ

$$\Sigma_{\alpha\beta} = 0 \tag{5.5}$$

2) A vector field F^α transforms as

$$F^\alpha \xrightarrow{\Lambda} \Lambda^\alpha_{\alpha'} F^{\alpha'} \tag{5.6}$$

so from (5.3) and (5.1) we get an expression for Σ in vector case in the form

$$[\Sigma_{\alpha\beta}]^\gamma{}_\delta = \delta^\gamma_\alpha \eta_{\beta\delta} - \delta^\gamma_\beta \eta_{\alpha\delta}. \tag{5.7}$$

3) For a spinor field it may be written as

$$[\Sigma_{\alpha\beta}]^\gamma{}_\delta = (1/4)[\gamma_\alpha, \gamma_\beta]^\gamma{}_\delta, \tag{5.8}$$

where γ are Dirac matrices.

Now we can generalize this concept to curved space-time.

5.2 Fields in curved space-time

Let us consider general coordinate transformations. We will define a general coordinate transformation as an arbitrary reparametrization of the coordinate system:

$$x'^{\mu} = x'^{\mu}(x^{\nu}) \tag{5.9}$$

Unlike Lorentz transformations, which are global space-time transformations, general coordinate transformations are local. Under reparametrizations, a scalar field transforms simply as follows:

$$\varphi'(x') = \varphi(x), \tag{5.10}$$

and vectors ∂_μ and dx^μ as:

$$\partial'_{\mu} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} \partial_{\nu}$$

$$dx'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} dx^{\nu} \tag{5.11}$$

As we can see, the transformation properties of (5.10) and (5.11) are based on the set of arbitrary real (4×4) matrices of the group $GL(4)$. Now we can give the abstract definition of covariant and contravariant tensors and write the general form of transformation

$$A'^{\mu_1 \ldots}_{\nu_1 \ldots}(x') = \frac{\partial x'^{\mu_1}}{\partial x^{\xi_1}} \ldots \frac{\partial x^{\lambda_1}}{\partial x'^{\nu_1}} \ldots A^{\xi_1 \ldots}_{\lambda_1 \ldots}(x) \tag{5.12}$$

Let us introduce a metric tensor $g_{\mu\nu}$ which allows us to write the four interval

$$ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu} \tag{5.13}$$

The tensor $g_{\mu\nu}$ transforms under the general coordinate transformation as genuine tensor. The derivative of the scalar field $\partial_\mu \varphi(x)$ is a genuine tensor, but the derivative of a vector is not. To create a vector from $\partial_\mu A_\nu(x)$ and $(\partial_\mu A^\nu(x))$ we introduce new fields, connections, that absorb unwanted terms. The covariant derivatives are written then as

$$\nabla_\mu A_\nu = \partial_\mu A_\nu + \Gamma^\lambda_{\mu\nu} A_\lambda$$

$$\nabla_\mu A^\nu = \partial_\mu A_\nu - \Gamma^\nu_{\mu\lambda} A^\lambda \tag{5.14}$$

From the restriction

$$\nabla g_{\mu\nu} = 0 \tag{5.15}$$

we get the expression for connection $\Gamma^\lambda_{\mu\nu}$:

$$\Gamma^\lambda_{\mu\nu} = (1/2) g^{\lambda\sigma} (\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}) \tag{5.16}$$

which are named the Christoffel symbols.

Another very important object in curved space-time is the Riemann curvature tensor $R^\xi_{\mu\nu\lambda}$, which arises from the commutator of the covariant derivatives

$$[\nabla_\mu, \nabla_\nu] A^\lambda = R^\xi_{\mu\nu\lambda} A_\xi \tag{5.17}$$

The Riemann tensor is written in the form

$$R^{\xi}_{\mu\nu\lambda} = \partial_\lambda \Gamma^{\xi}_{\nu\mu} - \partial_\nu \Gamma^{\xi}_{\mu\lambda} + \Gamma^{\xi}_{\sigma\nu}\Gamma^{\sigma}_{\mu\lambda} - \Gamma^{\xi}_{\lambda\sigma}\Gamma^{\sigma}_{\mu\nu} \tag{5.18}$$

Contracting the indicies of the Riemann tensor one can get the Ricci curvature tensor

$$R_{\mu\nu} = R^{\lambda}_{\mu\nu\lambda} \tag{5.19}$$

and scalar curvature

$$R = g^{\mu\nu} R_{\mu\nu} \tag{5.20}$$

The transformation properties of the volume element and the square root of the metric tensor are

$$d^4 x' = Det\left(\frac{\partial x'^{\mu}}{\partial x^{\nu}}\right) d^4 x$$

$$\sqrt{g'(x')} = Det\left(\frac{\partial x^{\mu}}{\partial x'^{\nu}}\right)\sqrt{g(x)} \tag{5.21}$$

From (5.21) we get that the product of these two is invariant

$$\sqrt{g(x)} d^4 x = inv. \tag{5.22}$$

Now we can construct the Einstein-Hilbert action as an invariant of the form

$$S_g = -(1/2k^2) \int d^4x \sqrt{g(x)} R, \tag{5.23}$$

and the action for a scalar field

$$S_\varphi = -(1/2) \int \sqrt{g(x)} \left(g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + m^2 \varphi^2 \right), \tag{5.24}$$

where the scalar matter couples to gravity via the interaction

$$\sqrt{g(x)} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi \sim g^{\mu\nu} T_{\mu\nu}.$$

The coupling of the gravitational field to the vector field is also straightforward

$$S_v = -(1/4) \int \sqrt{g(x)} g^{\mu\nu} g^{\lambda\xi} F^a_{\mu\lambda} F^a_{\nu\xi} \tag{5.25}$$

For the gauge-fixing term the generalization to curved space-time is expressed by

$$S_{gf} = -(1/2)\alpha^{-1} \int d^4x \sqrt{g(x)} \left(\nabla_\mu A^\mu \right)^2 \tag{5.26}$$

For the ghost term it is of the form:

$$S_{gh} = \int d^4x \sqrt{g(x)} \left(-\partial_\mu c^\alpha \partial^\mu \bar{c}^\alpha \right) \tag{5.27}$$

However, the coupling of gravity to spinor fields leads to a difficulty, because there are no finite dimensional spinorial representations of $GL(4)$. This prevents a naive incorporation of spinors into general relativity. The method we may use for these constructions involves vierbein formalism.

5.3 Spinors in General relativity

The vierbein (tetrad) formalism utilizes the fact that we can construct a flat tangent space to the curved manifold and introduce the spinorial representation of the Lorentz group in each point of the manifold in this tangent space. Spinors can then be defined at any point on the curved manifold only if they transform within the flat tangent space.

Let us erect normal coordinates $y^{\alpha}_{(X)}$ at each point X of the space-time manifold M. To preserve the connection with the previous sections we will label the flat tangent indices with letters α, β, γ, δ,... from the beginning of the Greek alphabet, as we did it before, and general coordinate transformation indices with letters λ, μ, ν, ξ,... from the end of the Greek alphabet.

Introduce the vierbein as the mixed tensor (matrix)

$$h^{\alpha}_{\mu}(X) = \left(\frac{\partial y^{\alpha}_{(X)}}{\partial x^{\mu}}\right)_{x=X} \qquad \alpha = 0, 1, 2, 3 \tag{5.28}$$

Note that the label α refers to the local inertial frame associated with normal coordinates $y^{\alpha}_{(X)}$ at the point X, while μ is associated with the general coordinate system

$\{x^\mu\}$.

The inverse of this matrix is given by $h^\mu_\alpha(X)$.

$$h^\alpha_\mu h^\mu_\beta = \delta^{ab} \tag{5.29}$$

The vierbein can be viewed as the "square root" of the metric tensor $g_{\mu\nu}$:

$$g_{\mu\nu} = h^\alpha_\mu h^\beta_\nu \eta_{\alpha\beta} \tag{5.30}$$

For general coordinate transformations $x^\mu = x^\mu(x^{\mu\prime})$ we can consider the effect of changing the x^μ while leaving the $y^\alpha_{(X)}$ fixed. Then the verbein transforms as

$$h^\alpha_\mu \rightarrow h'^\alpha_\mu = \frac{\partial x'^\mu}{\partial x^\mu} h^\alpha_\mu \tag{5.31}$$

We also can transform the $y^\alpha_{(X)}$ arbitrarily at each point X

$$y^\alpha_{(X)} \rightarrow y'^\alpha_{(X)} = \Lambda(X)^\alpha_\beta y^\beta_{(X)} \tag{5.32}$$

In this case $h^\alpha_\mu(X)$ transforms as a Lorentz covariant vector

$$h^\alpha_\mu(X) \rightarrow h'^\alpha_\mu(X) = \Lambda(X)^\alpha_\beta h^\beta_\mu(X) \tag{5.33}$$

which leaves the metric (5.30) invariant.

If a covariant vector A_μ is contracted into h^μ_α, the resulting object

$$A_\alpha = h^\mu_\alpha A_\mu \tag{5.34}$$

transforms as a collection of four scalars under a general coordinate transformations, while under local Lorentz transformations (5.1) it behaves as a vector.

Thus, by use of tetrads, one can convert general tensors into local, Lorentz-transforming tensors, shifting the additional space-time dependence into the tetrads.

Now we may construct the generally covariant Dirac equation.

We introduce a spinor $\psi(x)$ that is defined as a scalar under general coordinate transformations and an ordinary spinor under flat tangent space Lorentz transformation:

Coordinate transformations: $\psi \to \psi$

Lorentz transformations: $\psi \to D[\Lambda(x)]\psi$

It is important to note that we have introduced local Lorentz transformations in flat tangent space, so $\omega_{\alpha\beta}$ is a function of the space-time. This means that the derivative of the spinor is no longer a genuine tensor. Therefore we must introduce a connection field $\omega_\mu^{\alpha\beta}$ that allows us to gauge the Lorentz group. The covariant derivative for gauging the Lorentz group may be written as

$$\nabla_\mu \psi = (\partial_\mu + (1/2)\Sigma_{\alpha\beta}\omega_\mu^{\alpha\beta})\psi \tag{5.35}$$

Let $\{\gamma^\alpha\}$ be a set of Dirac matrices with

$$\{\gamma_\alpha, \gamma_\beta\} = 2\eta_{\alpha\beta} \tag{5.36}$$

in the tangential space-time.

The Dirac matrices γ^{α} can be contracted with vierbeins:

$$h^{\mu}_{\alpha}(x)\gamma_{\alpha} = \gamma_{\mu}(x) \tag{5.37}$$

Then

$$\{\gamma_{\mu}(x), \gamma_{\nu}(x)\} = 2g_{\mu\nu}(x) \tag{5.38}$$

In the result (5.33) and (5.35) the generally covariant Dirac equation is given by

$$(i\gamma^{\mu}(x)\nabla_{\mu,x} + m)\psi(x) = 0 \tag{5.39}$$

and hence the action for Dirac particle interacting with gravity is given by:

$$L = \sqrt{g}\bar{\psi}(x)(i\gamma^{\mu}(x)\nabla_{\mu,x} + m)\psi(x) \tag{5.40}$$

where $\sqrt{g} = Det(h^{\alpha}_{\mu})$.

We can construct a new, alternative, version of the curvature tensor by taking the commutator of two covariant derivatives:

$$[\nabla_{\mu}, \nabla_{\nu}]\,\psi = \frac{1}{2}R^{\alpha\beta}_{\mu\nu}\Sigma_{\alpha\beta}\psi \tag{5.41}$$

Written out, this curvature tensor is generally covariant in μ, ν, but flat in α, β:

$$R_{\mu\nu}^{\alpha\beta} = \partial_\mu \omega_\nu^{\alpha\beta} - \partial_\nu \omega_\mu^{\alpha\beta} + \omega_{\mu\lambda}^\alpha \omega_\nu^{\beta\lambda} - \omega_{\nu\lambda}^\alpha \omega_\mu^{\beta\lambda} \tag{5.42}$$

At this point, the spinor connection $\omega_{\mu\lambda}^\alpha$ is still an independent field. Covariant derivative of the object with two different kinds of indicies is written as:

$$\nabla_\mu A_\mu^\alpha = \partial_\mu A_\nu^\alpha + \Gamma_{\mu\nu}^\xi A_\xi^\alpha + \omega^\alpha{}_{\mu\beta} A_\nu^\beta \tag{5.43}$$

The external constraint

$$\nabla_\mu h_\mu^\alpha = \partial_\mu h_\nu^\alpha + \Gamma_{\mu\nu}^\xi h_\xi^\alpha + \omega^\alpha{}_{\mu\beta} h_\nu^\beta = 0 \tag{5.44}$$

helps to express the spin connection through veirbeins

$$\omega_\mu^{\alpha\beta} = \frac{1}{2} h^{\alpha\nu} (\partial_\mu h_\nu^\beta - \partial_\nu h_\mu^\beta)$$

$$+ \frac{1}{4} h^{\alpha\nu} h^{\beta\xi} (\partial_\xi h_{\nu\gamma} - \partial_\nu h_{\xi\nu}) h_\mu^\nu - (\alpha \leftrightarrow \beta) \tag{5.45}$$

After this preliminary work we can study properties of bosonic and fermionic fields in an external gravitational field.

Chapter 6

BOSE FIELDS

IN CURVED SPACE-TIME

As we know already the inclusion of the interaction with gravitational field (theory formulated in curved space-time) is accomplished by replacing of the partial derivative by the covariant derivative $\partial_\mu \to \nabla_\mu$. It is necessary to ensure that the Lagrangian is the scalar under the general coordinate transformation. Integration is performed over the invariant volume. This procedure, based on general -coordinate covariance, is called the minimal interaction for gravity and leads to the action (5.24).

However, general coordinate covariance does not forbid adding to the Lagrangian invariant terms which are vanishing in flat space-time. Such terms describe the non-minimal interaction with gravity. Therefore the theory under such consideration can be written in the form

$$S = \int d^4x \sqrt{-g} \left(L(\phi, \nabla_\mu \phi) + non - min. \;\; int. \right) \tag{6.1}$$

From dimensional analysis of R and ϕ conclude that the term, describing non-minimal interaction of matter field with gravitational field may be written in the form:

$$(1/2)\xi R \varphi^2 \tag{6.2}$$

where ξ is a dimensionless parameter (non-minimal coupling constant) [Fulling 1989].

If $m = 0$ and $\xi = 1/6$ then the action is invariant not only under general-coordinate transformation but also under conformal transformations
[Birrell & Davies 1982]

$$g^{'}_{\mu\nu}(x) = e^{2\chi(x)} g_{\mu\nu}, \quad \varphi^{'}(x) = e^{\chi(x)} \varphi(x) \tag{6.3}$$

where $\chi(x)$ is an arbitrary scalar field (parameter of the conformal transformation).

In the result we will have the action in the form

$$S_\varphi = -(1/2) \int d^4x \sqrt{-g} \varphi(x) \left(-\Box_x + m^2 + \xi R \right) \varphi(x) \tag{6.4}$$

where the d'Alembertian operator is $\Box_x = g_{\mu\nu}(x) \nabla^x{}_\mu \nabla^x{}_\nu = \partial_\mu \partial^\mu$.

The generating functional will be

$$Z[R, J] \propto (DetG)^{-1/2} \times$$

$$\times \exp\left\{-(i/2)\int d^4xg(x)^{1/2}\int d^4x^{'}g(x^{'})^{1/2}J(x)G(x,x^{'})J(x^{'})\right\} \tag{6.5}$$

where $G(x,x^{'})$ is the Green's function of the scalar field which is described by the equation:

$$g^{(1/2)}(x)\left(-\Box_x+m^2+\xi R\right)G(x,x^{'})=\delta(x-x^{'}) \tag{6.6}$$

So, for computation of the generating functional (6.5) we have to define the Green's function of the scalar field in curved space-time from the equation (6.6).

6.1 Momentum-space representation of the bosonic Green's function

In curved space-time we cannot solve the equation (6.6) and compute the Green's function $G(x,x^{'})$ for arbitrary points x and $x^{'}$ of the manifold. But we can do so in the particular case of the limit of coincidence ($x\simeq x^{'}$). This limit gives us a possibility to find the (Det) of the Green's functions to get the effective action. Thus we will treat the problem of the Green's functions calculations in the limit of coincidence.

Let us select a point of the space-time manifold $x^{'}$ and construct the tangential space at this point as we did in the previous section. Let any point x of the manifold have the normal coordinate $y^\alpha(x)$. This coordinate may be treated as a vector in the tangent space with the origin at the point $x^{'}$. In this tangent space the norm of the vector $y^\alpha(x)$ will be $(y,y)=y^\alpha(x)y^\beta(x)\eta_{\alpha\beta}$, and the metric of the manifold may be written as [Petrov 1969]

$$g_{\mu\nu}(x) = \eta_{\mu\nu} - \frac{1}{3}R_{\mu\alpha\nu\beta}y^{\alpha}y^{\beta} - \frac{1}{6}R_{\mu\alpha\nu\beta;\gamma}y^{\alpha}y^{\beta}y^{\gamma} + ... \tag{6.7}$$

and

$$g(x) = 1 - \frac{1}{3}R_{\alpha\beta}y^{\alpha}y^{\beta} - \frac{1}{6}R_{\alpha\beta;\gamma}y^{\alpha}y^{\beta}y^{\gamma}$$

$$+ \left(\frac{1}{18}R_{\alpha\beta}R_{\gamma\delta} - \frac{1}{90}R^{\nu}_{\alpha\beta\lambda}R^{\lambda}_{\gamma\delta\nu} - \frac{1}{20}R_{\alpha\beta;\gamma\delta}\right)y^{\alpha}y^{\beta}y^{\gamma}y^{\delta} + ... \tag{6.8}$$

where the coefficients are calculated at the $(y = 0)$ origin of the coordinate system.

The second derivative for the scalar field is

$$\nabla^{x\mu}\nabla^{x}_{\mu} = \eta^{\alpha\beta}\partial_{\alpha}\partial_{\beta} + \frac{1}{3}R_{\alpha}{}^{\delta}{}_{\beta}{}^{\gamma}y^{\alpha}y^{\beta}\partial^{x}_{\delta}\partial^{x}_{\gamma} - \frac{2}{3}R^{\alpha}_{\beta}y^{\beta}\partial^{x}_{\alpha} + ... \tag{6.9}$$

For our calculations it is convenient to express the Green's function $G(x,x^{'})$ as:

$$G(x,x^{'}) = g^{-1/4}(x)\Im(x,x^{'})g^{-1/4}(x^{'})$$

$$= g^{-1/4}(x)\Im(x,x^{'}) \tag{6.10}$$

Substituting (6.8),(6.9) and (6.10) into (6.6), we find

$$\eta^{\alpha\beta}\partial_{\alpha}\partial_{\beta}\Im - \left[m^{2} + \left(\xi - \frac{1}{6}\right)R\right]\Im - \frac{1}{3}R_{\alpha}{}^{\beta}y^{\alpha}\partial_{\beta}\Im$$

$$+\frac{1}{3}R_{\alpha}{}^{\beta}{}_{\gamma}{}^{\delta}y^{\alpha}y^{\gamma}\partial_{\beta}\partial_{\delta}\Im-\left(\xi-\frac{1}{6}\right)R_{;\alpha}y^{\alpha}\Im$$

$$+\left(-\frac{1}{3}R_{\alpha}{}^{\beta}{}_{;\gamma}+\frac{1}{6}R_{\alpha}\gamma^{;\beta}\right)y^{\alpha}y^{\gamma}\partial_{\beta}\Im$$

$$+\frac{1}{6}R_{\lambda}{}^{\gamma}{}_{\alpha}{}^{\zeta}{}_{;\beta}y^{\lambda}y^{\alpha}y^{\beta}\partial_{\gamma}\partial_{\zeta}\Im-\frac{1}{2}\left(\xi-\frac{1}{6}\right)R_{;\alpha\beta}y^{\alpha}y^{\beta}\Im+$$

$$\Big(-\frac{1}{30}R_{\alpha}{}^{\beta}R_{\beta\gamma}+\frac{1}{60}R_{\alpha}{}^{\beta}{}_{\gamma}{}^{\delta}R_{\beta\delta}$$

$$+\frac{1}{60}R^{\beta\chi\delta}{}_{\alpha}R_{\beta\chi\delta\gamma}-\frac{1}{120}R_{;\alpha\gamma}+\frac{1}{40}\Box R_{\alpha\gamma}\Big)\Im$$

$$+\Big(-\frac{3}{20}R^{\delta}{}_{\alpha;\beta\gamma}+\frac{1}{10}R_{\alpha\beta}{}^{;\delta}{}_{\gamma}-\frac{1}{60}R^{\chi}{}_{\alpha}{}^{\delta}{}_{\beta}R_{\chi\gamma}$$

$$+\frac{1}{15}R^{\chi}{}_{\alpha\lambda\beta}R_{\chi}{}^{\delta}{}_{\gamma}{}^{\lambda}\Big)y^{\alpha}y^{\beta}y^{\gamma}\partial_{\delta}\Im$$

$$+\left(\frac{1}{20}R^{\kappa}{}_{\alpha}{}^{\chi}{}_{\beta;\gamma\delta}+\frac{1}{15}R^{\kappa}{}_{\alpha\lambda\beta}R^{\lambda}{}_{\gamma}{}^{\chi}{}_{\delta}\right)y^{\alpha}y^{\beta}y^{\gamma}y^{\delta}\partial_{\kappa}\partial_{\chi}\Im=-\delta(y),\qquad(6.11)$$

where y^{α} are the coordinates of the point x and $\partial_{\alpha}\Im=(\partial/\partial y^{\alpha})$.

We have retained only terms with coefficients involving four derivatives of the metric. These contributions give the ultraviolet divergences that arise in the course of renormalization.

In normal coordinates with origin at $x^{'}$, $\Im(x,x^{'})$ is a function of y and $x^{'}$

$$\Im(x,x^{'}) = \Im(y,x^{'}) \tag{6.12}$$

where y belongs to the small region around $x^{'}$. In this way the equation (6.6) may be solved recursively. Namely, we will introduce the momentum space associated with the point $x^{'}$ $(y=0)$ by making the n-dimensional Fourier transformation:

$$\Im(x,x^{'}) = \int \frac{d^n k}{(2\pi)^n} \Im(k) \exp(iky) \tag{6.13}$$

where $ky = k_\alpha y^\alpha = \eta^{\alpha\beta} k_\alpha y_\beta$ and expanding $\Im(k)$ in a series:

$$\Im(k) = \Im_0(k) + \Im_1(k) + \Im_2(k) + ... \tag{6.14}$$

or

$$\Im_i(x,x^{'}) = \int \frac{d^n k}{(2\pi)^n} \Im_i(k) \exp(iky), \quad i = 0,1,2,... \tag{6.15}$$

where we will assume that the coefficients $\Im(k)$ have geometrical coefficients involving i derivatives of the mertic.

On dimensional grounds, $\Im_i(k)$ are the order $k^{-(2+i)}$, so that (6.14) is an asymptotic expansion of $\Im(k)$ in large k (small y).

Inserting (6.13) into (6.11) we get that the lowest order solution ($\sim O(y^2)$) is

$$\Im_0(k) = (k^2 + m^2)^{-1} \tag{6.16}$$

and

$$\Im_1(k) = 0 \tag{6.17}$$

The function $\Im_2(k)$ ($\sim O(y^4)$) may be found from

$$(\eta^{\alpha\beta}\partial_\alpha\partial_\beta - m^2)\Im_2 - \left(\xi - \frac{1}{6}\right) R\Im_0$$

$$-\frac{1}{3}R_\alpha{}^\beta y^\alpha \partial_\beta \Im_0 + \frac{1}{3}R_\alpha{}^\beta{}_\gamma{}^\delta y^\alpha y^\gamma \partial_\beta \partial_\delta \Im_0 = 0 \tag{6.18}$$

Using (6.13) one can get that the last two terms of (6.18) cancel each other.

In another way we can consider that $\Im_0$ is Lorentz invariant of the form

$$\Im(y) = y_\alpha y^\alpha = \eta^{\alpha\beta} y_\alpha y_\beta \tag{6.19}$$

Then, inserting (6.19) into (6.18) we get:

$$-\frac{1}{3}R_\alpha{}^\beta y^\alpha \partial_\beta \Im_0(y) + \frac{1}{3}R_\alpha{}^\beta{}_\gamma{}^\delta y^\alpha y^\gamma \partial_\beta \partial_\delta \Im_0(y) \equiv 0, \tag{6.20}$$

and

$$(\eta^{\alpha\beta}\partial_\alpha\partial_\beta - m^2)\Im_2(y) - \left(\xi - \frac{1}{6}\right) R\Im_0(y) = 0 \tag{6.21}$$

Therefore

$$\Im_2(k) = \left(\frac{1}{6} - \xi\right) R(k^2 + m^2)^{-2}. \tag{6.22}$$

The Lorentz invariance of $\Im_0(y)$ leads to further simplifications of (6.11). Namely, the contributions

$$\left(-\frac{1}{3}R_\alpha{}^\beta{}_{;\gamma} + \frac{1}{6}R_\alpha\gamma^{;\beta}\right) y^\alpha y^\gamma \partial_\beta\zeta\Im_0(y) + \frac{1}{6}R_\lambda{}^\gamma{}_\alpha{}^\zeta{}_{;\beta} y^\lambda y^\alpha y^\beta \partial_\gamma\partial_\zeta\Im_0(y) \equiv 0 \tag{6.23}$$

are eliminated.

In the same way

$$\left(-\frac{3}{20}R^\delta{}_{\alpha;\beta\gamma} + \frac{1}{10}R_{\alpha\beta}{}^{;\delta}{}_\gamma - \frac{1}{60}R^\chi{}_\alpha{}^\delta{}_\beta R_{\chi\gamma}\right.$$

$$\left. + \frac{1}{15}R^\chi{}_{\alpha\lambda\beta}R_\chi{}^\delta{}_\gamma{}^\lambda\right) y^\alpha y^\beta y^\gamma \partial_\delta\Im_0(y)$$

$$+ \left(\frac{1}{20}R^\kappa{}_\alpha{}^\chi{}_{\beta;\gamma\delta} + \frac{1}{15}R^\kappa{}_{\alpha\lambda\beta}R^\lambda{}_\gamma{}^\chi{}_\delta\right) y^\alpha y^\beta y^\gamma y^\delta \partial_\kappa\partial_\chi\Im_0(y) \equiv 0 \tag{6.24}$$

and we have the following equation for $\Im(y)$ to the fourth order in derivatives of the metric:

$$\eta^{\alpha\beta}\partial_\alpha\partial_\beta\Im(y) - \left[m^2 + \left(\xi - \frac{1}{6}\right) R\right] \Im(y)$$

$$- \left(\xi - \frac{1}{6}\right) R_{;\alpha} y^\alpha \Im(y)$$

$$- (1/2) \left(\xi - \frac{1}{6}\right) R_{;\alpha\beta} y^\alpha y^\beta \Im(y)$$

$$\left(-\frac{1}{30} R_\alpha{}^\beta R_{\beta\gamma} + \frac{1}{60} R_\alpha{}^\beta{}_\gamma{}^\delta R_{\beta\delta}\right.$$

$$\left.+\frac{1}{60} R^{\beta\chi\delta}{}_\alpha R_{\beta\chi\delta\gamma} - \frac{1}{120} R_{;\alpha\gamma} + \frac{1}{40} \Box R_{\alpha\gamma}\right) \Im(y) = -\delta(y) \tag{6.25}$$

Substitution of $\Im_2(k)$ (6.22) instead of $\Im_0(k)$ in the identity (6.20) does not change it, thus we can suggest that $\Im_2(k)$ is Lorentz invariant too and it can be wtitten as $\Im_2(y) \sim (y^\alpha y_\alpha)^2$. The equation (6.11) is simplified to

$$\left[k^2 + m^2 + \left(\xi - \frac{1}{6}\right) R + i \left(\xi - \frac{1}{6}\right) R_{;\alpha} \partial^\alpha\right] \Im(k) +$$

$$+ \left[-(1/2) \left(\xi - \frac{1}{6}\right) R_{;\alpha\beta} - \frac{1}{30} R_\alpha{}^\beta R_{\beta\gamma} + \frac{1}{60} R_\alpha{}^\beta{}_\gamma{}^\delta R_{\beta\delta}\right.$$

$$\left.+\frac{1}{60} R^{\beta\chi\delta}{}_\alpha R_{\beta\chi\delta\gamma} - \frac{1}{120} R_{;\alpha\gamma} + \frac{1}{40} \Box R_{\alpha\gamma}\right] \partial^\alpha \partial^\beta \Im(k) = 1 \tag{6.26}$$

where

$$\partial^\alpha \Im(k) = \partial \Im(k) / \partial k_\alpha$$

Making a further recurrent process with this equation we get

$$\Im_3(k) = 0 \tag{6.27}$$

and

$$\Im_4(k) = i\left(\frac{1}{6} - \xi\right) R_{;\alpha}(k^2+m^2)^{-1}\partial^\alpha(k^2+m^2)^{-1} +$$

$$+\left(\frac{1}{6} - \xi\right)^2 R^2(k^2+m^2)^{-3} +$$

$$+ a_{\alpha\beta}(k^2+m^2)^{-1}\partial^\alpha\partial^\beta(k^2+m^2)^{-1} \tag{6.28}$$

where

$$a_{\alpha\beta} = (1/2)\left(\xi - \frac{1}{6}\right) R_{;\alpha\beta} + \frac{1}{30} R_\alpha{}^\beta R_{\beta\gamma} - \frac{1}{60} R_\alpha{}^\beta{}_\gamma{}^\delta R_{\beta\delta}$$

$$- \frac{1}{60} R^{\beta\chi\delta}{}_\alpha R_{\beta\chi\delta\gamma} + \frac{1}{120} R_{;\alpha\gamma} - \frac{1}{40} \Box R_{\alpha\gamma} \tag{6.29}$$

Now we can write the equation for the Green 's function in a convenient form.

Let us introduce useful equations:

$$(k^2+m^2)^{-1}\partial^\alpha(k^2+m^2)^{-1} \equiv (1/2)\partial^\alpha(k^2+m^2)^{-2}$$

and

$$(k^2+m^2)^{-1}\partial^\alpha\partial^\beta(k^2+m^2)^{-1} \equiv (1/3)\partial^\alpha\partial^\beta(k^2+m^2)^{-2}$$

$$-\,(2/3)\eta_{\alpha\beta}(k^2+m^2)^{-3} \qquad (6.30)$$

Using (6.30) we may write $\Im(k)$ as

$$\Im(k) = (k^2+m^2)^{-1} + \left(\frac{1}{6}-\xi\right) R(k^2+m^2)^{-2}$$

$$(i/2)\left(\frac{1}{6}-\xi\right) R_{;\alpha}(k^2+m^2)^{-2}\partial^\alpha(k^2+m^2)^{-1}$$

$$+\,(1/3)a_{\alpha\beta}\partial^\alpha\partial^\beta(k^2+m^2)^{-2}$$

$$+\left[\left(\frac{1}{6}-\xi\right)^2 R^2 - (2/3)a_\lambda^\lambda\right](k^2+m^2)^{-3} \qquad (6.31)$$

Inserting the last equation into (6.13) we get:

$$\Im(x^{'},y) = \int \frac{d^n k}{(2\pi)^n}\exp(iky)\left[1+\gamma_1(x^{'},y)\left(-\frac{\partial}{\partial m^2}\right)\right.$$

$$\left.+\gamma_2(x^{'},y)\left(-\frac{\partial}{\partial m^2}\right)^2\right](k^2+m^2)^{-1} \qquad (6.32)$$

where, to the fourth order in derivatives of the metric, the coefficients are:

$$\gamma_1(x', y) = \left(\frac{1}{6} - \xi\right) R + (1/2)\left(\frac{1}{6} - \xi\right) R_{;\alpha} y^\alpha - (1/3) a_{\alpha\beta} y^\alpha y^\beta;$$

$$\gamma_2(x', y) = (1/2)\left(\frac{1}{6} - \xi\right)^2 R^2 - (1/3) a_\lambda^\lambda. \tag{6.33}$$

The expression for the Green's function (6.10) will be then [Bunch & Parker 1979]

$$G(x', y) = g^{-1/2}(y) \int \frac{d^n k}{(2\pi)^n} \exp(iky) \times$$

$$\times \sum_{j=0}^{2} \gamma_j(x', y) \left(-\frac{\partial}{\partial m^2}\right)^j (k^2 + m^2)^{-1} \tag{6.34}$$

The equation (6.34) is very important for further calculations.

6.2 The Green's function and the Schwinger-DeWitt method

We have found already that the geometrical quantities enter directly into the structure of Green's functions for arbitrary fields through the covariant d'Alembertian operator and non-minimal connection. Now we will treat the equation for the Green's function with the Schwinger-DeWitt method developed for flat space-time in the previous section.

Let us multiply the equation (6.6) on the left side by $g^{(1/4)}(x)$ and on the right by $g^{(1/4)}(x')$, and introducing

$$\Im(x,x^{'}) = g^{(1/4)}(x)G(x,x^{'})g^{(1/4)}(x^{'}) \tag{6.35}$$

we will rewrite this equation in the form

$$\left(-\partial_\mu\partial^\mu + m^2 + \xi R\right)\Im(x,x^{'}) = \delta(x,x^{'}), \tag{6.36}$$

where $\delta(x,x^{'}) = g^{-1/2}(x)\delta(x-x^{'})$ is scalar with respect to general coordinate transformation, and product of δ -functon is

$$(1,\delta(x,x^{'})) = \int d^4x\delta(x,x^{'}) = 1 \tag{6.37}$$

For simplicity we put $\xi = 0$ and get;

$$\left(-\Box_x + m^2)\right)\Im(x,x^{'}) = \delta(x,x^{'}), \tag{6.38}$$

or, in operator form

$$\hat{F}\Im = 1 \tag{6.39}$$

where F is matrix operator.

Let us introduce the representation for $\Im$ in the form

$$\Im(x,x^{'};s) = i < x|\int_0^\infty ds\exp(is\hat{F})|x^{'} >= i\int_0^\infty dsf(x,x^{'};s)\exp(-im^2s) \tag{6.40}$$

We can get from (6.38), that the function $f(x, x^{'}; s)$ is the solution of the equation

$$\Box_x f(x, x^{'}; s) = i\frac{\partial}{\partial s} f(x, x^{'}; s) \tag{6.41}$$

where information about space-time structure is included in the d'Alembertian.

We may turn this equation into an elliptic one by rewriting the equation for amplitude with the replacement $x^{(0)} = ix^{(4)}$ and $s = it$. We will have

$$\frac{\partial}{\partial t} f(x, x^{'}; t) = \tilde{\Box}_x f(x, x^{'}; t) \tag{6.42}$$

One can write the solution of the (6.40) as a simple expansion of the solution for flat space-time. This solution is

$$f(x, x^{'}; \bar{s}) = (4\pi\bar{s})^{-n/2} \exp(-|x - x^{'}|^2/4\bar{s}) \tag{6.43}$$

Returning to the initial variables we get

$$f(x, x^{'}; s) = (4\pi is)^{-n/2} \exp(-|x - x^{'}|^2/4is) \tag{6.44}$$

In curved space-time we may expand this solution to a local asymptotic expansion (for $x \simeq x^{'}$ and $s \simeq 0$)

$$f(x, x^{'}; s) \sim (4\pi is)^{(-n/2)} \exp(-\sigma(x, x^{'})/2is) \sum_{j=0}^{\infty} \gamma_j(x, x^{'})(is)^j \tag{6.45}$$

where $\sigma(x, x^{'})$ is the so-called geodesic interval (half square of geodesic distance between points x and $x^{'}$). In particular

$$f(x, x; s) \sim (4\pi is)^{(-n/2)} \sum_{j=0}^{\infty} \gamma_j(x)(is)^j \tag{6.46}$$

The explicit form of $f_j(x, x^{'})$ can be calculated recursively [Brown 1979] [DeWitt 1975], [Fulling 1989]

In the limit of coincidence $x \to x^{'}$ one finds:

$$\gamma_0(x^{'}) = 1; \quad \gamma_1(x^{'}) = (1/6 - \xi)\, R;$$

and

$$\gamma_2(x^{'}) = (1/6 - \xi)^2\, R^2 - (1/3) a^{\lambda}_{\lambda}. \tag{6.47}$$

From the equations (6.40), (6.45) and (6.35) we get the explicit expression for Green's function in the Schwinger-DeWitt representation:

$$G_{SD}(x, x^{'}) = \frac{i\Delta^{1/2}(x, x^{'})}{(4\pi)^{n/2}} \int_0^{\infty} ids(is)^{(-n/2)} \times$$

$$\times \exp\left(-ism^2 - \sigma(x, x^{'})/2is\right) \sum_{j=0}^{\infty} \gamma_j(x, x^{'})(is)^j \tag{6.48}$$

where

$$\Delta(x,x^{'}) = -g(x)^{-1/2} det[\partial_\alpha \partial_\beta \sigma(x,x^{'})] g(x)^{-1/2} \tag{6.49}$$

is the Van Vleck determinant (in normal coordinates about $x^{'}$ this determinant is reduced to $g^{-1/2}(y)$ (6.34)).

6.3 Connection between the two methods

Let us put

$$(k^2+m^2)^{-1} = \int_0^\infty ids \exp[-is(k^2+m^2)] \tag{6.50}$$

Then integration over the momentum in (6.34) leads to

$$\int \frac{d^n k}{(2\pi)^n} \exp[-is(k^2+m^2)+iky] = i(4\pi is)^{n/2} \exp(-ism^2 - \sigma/2is) \tag{6.51}$$

The resulting equation for the Green's function will be

$$G(y,x^{'}) = \frac{i}{(4\pi)^{n/2}} g(y)^{-1/2} \int_0^\infty \frac{ds}{(is)^{n/2}} \times$$

$$\times \exp\left[-is(k^2+m^2)+iky\right] F(x^{'},y,is) \tag{6.52}$$

where

$$F(x^{'},y,is) = \sum_{j=0}^{2} \gamma_j(x^{'},y)(is)^j$$

Comparision with the expression of the Green's function in the form of (6.48) gives $\Delta(x, x') = g^{-1/2}(y)$, and

$$\gamma_j(x', y) = \gamma_j(x, x') \tag{6.53}$$

In this chapter we considered two methods for computing the Green's function of a scalar field in curved space-time and we prepared the basis for future finite temperature calculations.

Chapter 7

FINITE TEMPERATURE BOSONS

IN CURVED SPACE-TIME

In the previous section we developed a mathematical formalism which is a convenient tool for the description of thermal Bose gas in curved space-time. In this section we can consider the ensemble of bosons interacting with gravity at finite temperature.

Let a total system ("matter and gravity") be described by the action

$$S_{tot} = S_g + S_m \tag{7.1}$$

The gravitational action is

$$S_g = \int d^4x \sqrt{g(x)} L_g \tag{7.2}$$

with Lagrangian

$$L_g = \frac{1}{16\pi G_0}(R - 2\Lambda_0) + \alpha_0 R^2 + \beta_0 R_{\alpha\beta}R^{\alpha\beta} + \gamma_0 R_{\alpha\beta\chi\delta}R^{\alpha\beta\chi\delta} \tag{7.3}$$

and the action for a matter field is

$$S_m = \int d^4x\sqrt{g(x)}L_{eff,m} \tag{7.4}$$

To write the action for the matter field we use a generating functional $Z[J]$

$$Z[0] = \int D\varphi \exp\left(-(i/2)\int d^4x\sqrt{g(x)}\varphi(x)(-\Box_x + m^2 + \xi R)\varphi(x)\right)$$

$$\propto DetG(x,x^{'})^{1/2}\exp\left(-(i/2)(J(x),G(x,y)J(y))\right) \tag{7.5}$$

Then the functional $W[0] = -i\ln Z[0]$ will be

$$W[0] = -(i/2)\ln DetG(x,x^{'})$$

$$= -(i/2)\int\limits_0^\infty ids(is)^{-1}\mathrm{tr}f(x,x^{'},is)\exp(-im^2 s) \tag{7.6}$$

We may connect the Green's function and heat kernel with the equation

$$G_{SD}(x,x^{'}) = \int\limits_0^\infty ids f(x,x^{'},is)\exp(-im^2 s) \tag{7.7}$$

and write (7.6) in the following form

$$W[0] = -(i/2)\int d^4x\sqrt{g(x)}\int\limits_{m^2}^{\infty} dm^2 \mathrm{tr} G_{SD}(x,x^{'}) \tag{7.8}$$

From the equation (7.8) we find an important expression for the effective Lagrangian of matter field

$$L_{eff,m} = (-i/2)\int d^4x\sqrt{g(x)}\int\limits_{m^2}^{\infty} dm^2 \mathrm{tr} G(x,x^{'}) \tag{7.9}$$

Then the effective action will be:

$$S_{eff} = \int d^4x\sqrt{g(x)} L_{eff}(x)$$

$$= \int d^4x\sqrt{g(x)}\left(L_g(x) + (-i/2)\int\limits_{m^2}^{\infty} dm^2 \mathrm{tr} G(x,x^{'})\right) \tag{7.10}$$

In order to apply the usual formalism of finite temperature quantum field theory, we will assume that the space-time manifold $M_{4)}$ is a static manifold with topology $S^1 \times M_{3)}$ where S^1 refers to time coordinate and $M_{3)}$ is the spatial, three dimensional section of $M_{4)}$. We will choose $M_{3)}$ without boundaries [Kennedy et al. 1980], then no surface terms will appar in the induced action.

The heat kernel may be expressed as the sum of zero-temperature images [Denardo & Spalucci 1983, Dowker & Kennedy 1978, Dowker & Critchley 1977]

$$f(x,x,is) = \sum_{n=-\infty}^{\infty} \frac{\exp[-\beta^2 n^2/4is]}{(4\pi is)^{1/2}} f_{3)}(\tilde{x},\tilde{x},is) \tag{7.11}$$

where the sum goes from periodic restrictions and $f_{3)}(\tilde{x},\tilde{x},is)$ is the solution of the three-dimensional equation

$$\left(i\frac{\partial}{\partial s} - \Box_x + \xi R\right) f_{3)}(\tilde{x},\tilde{x},is) = 0 \tag{7.12}$$

This solution is written in the form of a series

$$f_{3)}(\tilde{x},\tilde{x},is) = \sum_{n=0}^{\infty} \gamma_j(R)(is)^j \tag{7.13}$$

Then from the equation (7.7) in the limit of coincidence $(x \to x^{'})$

$$\lim_{x\to x^{'}} G(x,x^{'}) = \lim_{x\to x^{'}} \int_0^{\infty} ids f(x,x^{'},is) \exp\{-im^2 s\}. \tag{7.14}$$

we get the finite temperature Green's function in the Schwinger-DeWitt representation:

$$\lim_{x\to x^{'}} G^{\beta}{}_{SD}(x,x^{'})$$

$$= \frac{i}{(4\pi)^{3/2}} \sum_{n=-\infty}^{\infty} \sum_{j=0}^{\infty} \gamma_j(x^{'}) \int_0^{\infty} ids(is)^{j-3/2} \exp(-ism^2 - n^2\beta^2/4is) \tag{7.15}$$

Selecting the temperature independent part $(n = 0)$ we find

$$\lim_{x \to x^{'}} G^{\beta}{}_{SD}(x, x^{'}) = G_{SD}(x^{'}, x^{'}) + G_{x^{'}}(\beta) \tag{7.16}$$

where the finite temperature contribution $G_{x^{'}}(\beta)$ is

$$G_{x^{'}}(\beta) = \frac{i}{(4\pi)^2} 2 \sum_{j=0}^{\infty} \sum_{n=1}^{\infty} \gamma_j(x^{'}) \times$$

$$\times \int_0^{\infty} ids(is)^{j-2} \exp(-ism^2 - n^2\beta^2/4is) \tag{7.17}$$

and $G_{SD}(x^{'}, x^{'})$ is the limit $(x \to x^{'})$ of the Green's function in the Schwinger-DeWitt representation (6.48).

Summation in (7.17) may be done with (3.27) and (3.31).

In the result we find

$$G_{x^{'}}(\beta) = \frac{i}{(2\pi)^2} \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \gamma_j(x^{'}) (\beta n/2m)^{j-3} K_{j-3}(\beta m n) \tag{7.18}$$

Total Lagrangian may be written in the form;

$$L_{eff}(\beta) = \left(L_g(x) + (-i/2) \int_{m^2}^{\infty} dm^2 \mathrm{tr} G_{SD}(x, x^{'}) \right) + (-i/2) \int_{m^2}^{\infty} dm^2 G_{x^{'}}(\beta) \tag{7.19}$$

The first two terms give the same geometric structure, and after renormalizations we will have the Lagrangian $\tilde{L}_g$, the third one is temperature contribution $f(\beta)$.

$$L_{eff}(\beta) = \tilde{L}_g - f(\beta) \tag{7.20}$$

where

$$\tilde{L}_g = \left(L_g(x) + (-i/2) \int\limits_{m^2}^{\infty} dm^2 \mathrm{tr} G_{SD}(x, x^{'}) \right) \quad (7.21)$$

and

$$f(\beta) = (i/2) \int\limits_{m^2}^{\infty} dm^2 G_{x'}(\beta) \quad (7.22)$$

The last expression can be written as series

$$f(\beta) = \sum_{j=0}^{3} \gamma_j(R) b_j(\beta m) \quad (7.23)$$

where

$$b_0(\beta m) = -\frac{m^2}{2\pi^2\beta^2} \sum_{n=1}^{\infty} (1/n^2) K_2(\beta mn)$$

$$b_1(\beta m) = -\frac{2m}{4\pi^2\beta} \sum_{n=1}^{\infty} (1/n) K_1(\beta mn)$$

$$b_2(\beta m) = -\frac{2}{8\pi^2} \sum_{n=1}^{\infty} K_0(\beta mn) \quad (7.24)$$

Using the results of Appendix: (XXII.28), (XXII.30), (XXII.32), we may rewrite (7.23) in the form of series [Kulikov & Pronin 1993]

$$f(\beta) = b_0(\beta m) + \gamma_1(R) b_1(\beta m) + \gamma_2(R^2) b_2(\beta m) \quad (7.25)$$

where $\gamma_j(R)$ are (6.33), coefficient $b_0(\beta m)$ is

$$b_0 = (1/\beta) \int \frac{d^3k}{(2\pi)^3} \ln(1 - \exp(-\beta\epsilon));$$

and

$$b_j(\beta m) = \left(-\frac{\partial}{\partial m^2}\right)^j b_0(\beta m) \tag{7.26}$$

The expression $b_0(\beta m)$ in (7.25) is the density of Helmholtz free energy in flat space and the following ones are created by corrections which are connected with the interaction of the heat bosons with gravity, so the equation (7.23) describes the Helmholtz free energy of a Bose gas in curved space time.

Chapter 8

FERMI FIELDS

IN CURVED SPACE-TIME

8.1 Momentum-space representation for the Green's function of a fermion

Now we will get the Green's function of fermions in curved space-time to apply the above results for computation of the effective action of the system "matter field + gravitational field".

As we know already, the fermionic action with Lagrangian (5.40) is written as

$$S_\psi = (i/2) \int d^4x \sqrt{g(x)} \bar{\psi}(x) \left(i\gamma^\mu \nabla_{\mu,x} + m \right) \psi(x) \tag{8.1}$$

The generating functional then is

$$Z[\bar{\eta},\eta] = \int D\bar{\psi}D\psi \exp\left[iS_\psi + i\left(\bar{\eta},\psi\right) + i\left(\bar{\psi},\eta\right)\right]$$

$$\propto (DetG_F)^{-1/2} \exp\left[-i\left(\bar{\eta}(x), S_F(x,y)\eta\right)\right] \tag{8.2}$$

where the scalar product includes the square root of the metric.

Connection between the bi-spinor G_F and Fermionic Green's function S_F is [Birrell & Davies 1982]:

$$S_F(x,y) = (i\gamma^\mu D_\mu + m)G_F(x,y) \tag{8.3}$$

and the Green's function satisfies the equation

$$(i\gamma^\mu D_\mu + m)S_F(x,y) = -g(x)^{-1/2}\delta(x-y)\hat{1} \tag{8.4}$$

To solve this equation with momentum space methods we introduce Riemann normal coordinates [Petrov 1969] and write the spin connection in the form [Panangaden 1981]

$$\Gamma_\mu(y) = (1/16)[\gamma_\alpha,\gamma_\beta]R^{\alpha\beta}{}_{\mu\nu}y^\nu + O(y^2) \tag{8.5}$$

and the vierbein field as

$$h^\alpha_\mu(y) = \delta^\alpha_\mu - (1/6)\eta^{\alpha\nu}R_{\mu\nu\beta\zeta}y^\beta y^\zeta + O(y^3) \tag{8.6}$$

The expression for $\gamma_\mu(x^{'})$ may be written as

$$\gamma_\mu(x^{'}) = h_\mu^\alpha(y)\gamma_\alpha = \delta_\mu^\alpha\gamma_\alpha - (1/6)\eta^{\alpha\nu}R_{\mu\nu\beta\zeta}y^\beta y^\zeta\gamma_\alpha + O(y^3) \tag{8.7}$$

The spinor derivative appearing in Dirac's equation is written as

$$\gamma_\mu D_\mu = \gamma_\mu(x^{'})(\partial_\mu - \Gamma_\mu)$$

$$= \gamma^\mu\partial_\mu + (1/6)R^\mu{}_\beta{}^\nu{}_\zeta y^\beta y^\zeta\partial_\mu -$$

$$- (1/16)\gamma^\mu[\gamma_\alpha, \gamma_\beta]R^{\alpha\beta}{}_{\mu\nu}y^\nu \tag{8.8}$$

The Fourier transform of $S(x^{'}, y)$ is

$$S(x^{'}, y) = \int \frac{d^nk}{(2\pi)^n}\exp(iky)S(k) \tag{8.9}$$

Let

$$S(k) = S_0(k) + S_1(k) + S_2(k) + ... \tag{8.10}$$

be an asymptotic representation of $S(k)$ for large k.

The values $S_i(k)$ are asymptotic variables of the order $k^{-(1+i)}$. They may be found with recursion procedure from the equation

$$\Big[(i\gamma^{\mu}\partial_{\mu}+m)+(1/6)R^{\mu}{}_{\beta}{}^{\nu}{}_{\zeta}y^{\beta}y^{\zeta}\partial_{\mu}-$$

$$-(1/16)\gamma^{\mu}[\gamma_{\alpha},\gamma_{\beta}]R^{\alpha\beta}{}_{\mu\nu}y^{\nu}+...\Big]S(x^{'},y)=\delta(y) \tag{8.11}$$

In this case the momentum space representation of a propagator of a fermion will be

$$S(x^{'},y)=\int\frac{d^{n}k}{(2\pi)^{n}}\exp(iky)\left[\frac{(\gamma\cdot k+m)}{k^{2}+m^{2}}+(1/4)R\frac{(\gamma\cdot k+m)}{(k^{2}+m^{2})^{2}}-\right.$$

$$\left.-(2/3)R_{\alpha\beta}k^{\alpha}k^{\beta}\frac{(\gamma\cdot k+m)}{(k^{2}+m^{2})^{3}}+(i/8)R^{\alpha\beta}{}_{\mu\nu}\frac{\gamma^{\mu}[\gamma_{\alpha},\gamma_{\beta}]k^{\nu}}{(k^{2}+m^{2})^{2}}+...\right] \tag{8.12}$$

The momentum space solution of the equation for the bi-spinor

$$\left(\Box_{x}+1/4R-m^{2}\right)G_{F}(x,x^{'})=-g(x)^{-1/2}\delta(x-x^{'})\hat{1} \tag{8.13}$$

where $\Box_{x}=D^{\mu}_{x}D_{\mu,x}$ is the covariant d'Alembertian of spinor field, may be obtained with the same momentum space methods as in chapter VI [Bunch & Parker 1979].

The result can be written in the form of the following expression

$$G_{F}(x,x^{'})=G_{F}(x^{'},y)$$

$$=g^{-1/2}(y)\int\frac{d^{n}k}{(2\pi)^{n}}\sum_{j=0}^{2}\hat{\alpha}_{j}(x^{'},y)\left(-\frac{\partial}{\partial m^{2}}\right)^{j}(k^{2}+m^{2})^{-1} \tag{8.14}$$

where the geometrical coefficients $\hat{\alpha}_{j}(x^{'},y)$ in the limit of coincidence $(x\rightarrow x^{'})$ are

$$\hat{\alpha}_0(x', y) = \hat{1};$$

$$\hat{\alpha}_1(x', y) = (1/12)R \cdot \hat{1};$$

and

$$\hat{\alpha}_2(x', y) = \Big(-(1/120)R_{\mu}{}^{;\mu} + (1/288)R^2-$$

$$-(1/180)R_{\mu\nu}R^{\mu\nu} + (1/180)R_{\mu\nu\sigma\tau}R^{\mu\nu\sigma\tau}) \cdot \hat{1}$$

$$+ (1/48)\Sigma_{[\alpha,\beta]}\Sigma_{[\gamma,\delta]}R^{\alpha\beta\lambda\xi}R\gamma_{\lambda}{}^{\delta}{}_{\xi} \tag{8.15}$$

where $\Sigma_{[\alpha,\beta]} = (1/4)[\gamma_\alpha, \gamma_\beta]$

8.2 The bi-spinor function in the Schwinger-DeWitt representation

In analogy with the scalar field we can rewrite equation (8.14) in the Schwinger-DeWitt representation. Since

$$(k^2 + m^2)^{-1} = \int_0^\infty ids \exp\Big[-is(k^2 + m^2)\Big] \tag{8.16}$$

the equation (8.14) will be

$$G_F(x,x^{'}) = \frac{i\Delta^{(1/2)}(x,x^{'})}{(4\pi^{n/2})} \int_0^{\infty} \frac{ids}{(is)^{n/2}} \exp\left[-is(k^2+m^2)\right] F(x,x^{'};s) \qquad (8.17)$$

where

$$F(x,x^{'};s) = \sum_{n=0}^{\infty} \hat{\alpha}_n(x,x^{'})(is)^n \qquad (8.18)$$

and coefficients $\hat{\alpha}_n(x,x^{'})$ are defined by (8.15)

As in the scalar case the determinant $\Delta^{(1/2)}(x,x^{'})$ is defined by the equation

$$\Delta^{(1/2)}(x,x^{'}) = g^{-1/2}(y) \qquad (8.19)$$

Chapter 9

FINITE TEMPERATURE FERMIONS

IN CURVED SPACE-TIME

9.1 The Helmholtz free energy of a Fermi gas in curved space-time

After we constructed the free energy for a thermal scalar field we can consider a thermal fermi field. Let the total Lagrangian of the system of "gravity + fermionic matter" be

$$S_{tot} = S_g + S_m \tag{9.1}$$

where S_g is (7.2) and

$$S_m = \int d^4x \sqrt{g(x)} L_{eff,\psi} \tag{9.2}$$

To write the action for a spinor field we will use the functional method for calculation of the generating functional from (9.2). Making the same procedure as in (7.6)-(7.8), write

$$W[0] = i \ln Z[0] = (i/2) \ln DetG_F(x, x^{'})$$

$$= \int_0^\infty ids(is)^{-1} \mathrm{tr} \hat{f}(x, x^{'}, is) \exp(-im^2 s)$$

$$= (i/2) \int d^4x \int_{m^2}^\infty dm^2 \mathrm{tr} G_F(x, x^{'}) \tag{9.3}$$

where the Green's function is expressed by

$$G_F(x, x^{'}) = \int_0^\infty ids(is)^{-1} \hat{f}(x, x^{'}, is) \exp(-im^2 s) \tag{9.4}$$

and $\hat{f}(x, x^{'}, is)$ is the heat kernel.

The kernel $\hat{f}(x, x^{'}, is)$ is (in the limit $(x \to x^{'})$) the sum of zero-images [Denardo & Spalucci 1983] antiperiodic in the imaginary time

$$\hat{f}(x, x^{'}, is) = \sum_{n=-\infty}^{\infty} \frac{\exp[-\beta^2 (n - 1/2)^2 / 4is]}{(4\pi is)^{1/2}} \hat{f}_{3)}(\tilde{x}, \tilde{x}, is) \tag{9.5}$$

where an equation for $\hat{f}_{3)}(x, x^{'}, is)$ is

$$\left(i\frac{\partial}{\partial s} - \Box_{3)} - (1/4)R\right)\hat{f}_{3)}(\tilde{x}, \tilde{x}, is) = 0 \tag{9.6}$$

and $\Box_{3)}$ is the covariant d'Alembertian on $M_{3)}$.

The solution of (9.6) is the series

$$\hat{f}_{3)}(\tilde{x}, \tilde{x}, is) = \sum_{j=0}^{\infty} \hat{\alpha}_j(x^{'})(is)^j \tag{9.7}$$

where the coefficients $\hat{\alpha}_j(x^{'})$ are determined by (8.15).

From the equation (9.4) we get the finite temperature Green's function

$$\lim_{x \to x^{'}} G^{\beta}_{F,SD}(x, x^{'}) = G_{F,SD}(x^{'}, x^{'}) + G_F(\beta) \tag{9.8}$$

where $G_{F,SD}(x^{'}, x^{'})$ is (8.17) in the limit $(x = x^{'})$ and

$$G_F(\beta) = \frac{i}{(4\pi)^3/2}\sum_{j=0}^{\infty}\sum_{n=-\infty}^{\infty}\hat{\alpha}_j(R)\times$$

$$\times\int_0^{\infty} ids(is)^{j-3/2}\exp\left[-ism^2 - (\beta^2/4is)(n+1/2)^2\right] \tag{9.9}$$

Summation with respect to (3.27) with $z = 1/2$ gives

$$G_F(\beta) = \frac{i}{(4\pi)^{3/2}}\sum_{j=0}^{\infty}\sum_{n=1}^{\infty}\hat{\alpha}_j(R)\times \tag{9.10}$$

$$\times 2 \int_0^\infty ids(is)^{j-2}(-1)^n \exp\left[-ism^2 - (n^2\beta^2/4is)\right] \tag{9.11}$$

After integration over the proper time (s) (3.31) we get the following expression for the finite temperature contribution in the Green's function of a fermion

$$G_F(\beta) = \frac{i}{(4\pi)^2} \sum_{j=0}^{\infty} \sum_{n=1}^{\infty} \hat{\alpha}_j(R)(-1)^j(\beta n/2m)^{j-3} K_{j-3}(\beta mn) \tag{9.12}$$

The total action of the system will be

$$L_{eff}(\beta) = \tilde{L}_g(R) - f_F(\beta, R) \tag{9.13}$$

where $\tilde{L}(R)_g$ is the temperature independent Lagrangian

$$\tilde{L}(R)_g = L_g + (i/2)\mathrm{tr} \int_{m^2}^{\infty} dm^2 G_{F,SD}(x, x^{'}), \tag{9.14}$$

and the finite temperature contribution is expressed in the form of a series:

$$f_F(\beta, R) = (-i/2)\mathrm{tr} \int_{m^2}^{\infty} dm^2 G_F(\beta) = \sum_{j=0}^{\infty} \alpha_j(R) f_j(\beta m) \tag{9.15}$$

with coefficients

$$\alpha_j(R) = (1/2s)\mathrm{tr}\hat{\alpha_j(R)}, \tag{9.16}$$

and

$$f_0(\beta m) = \frac{m^2 \cdot 2s}{2\pi^2\beta^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} K_2(\beta mn)$$

$$f_1(\beta m) = \frac{m \cdot 2s}{4\pi^2\beta} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} K_1(\beta mn)$$

$$f_2(\beta m) = \frac{2s}{8\pi^2} \sum_{n=1}^{\infty} (-1)^n K_0(\beta mn) \tag{9.17}$$

The finite, temperature dependent contribution $f(\beta, R)$ represents the density of Helmholtz free energy in curved space-time. Using the integral representation for the series of modified Bessel functions (XXII.10),(XXII.12),(XXII.14) one can write it as [Kulikov & Pronin 1995]:

$$f_F(\beta, R) = f_0(\beta m) + \alpha_1(R) f_1(\beta m) + \alpha_2(R) f_2(\beta m) + \ldots, \tag{9.18}$$

where the first term is the standard form of the Helmholtz free energy in Euclidean space

$$f_0(\beta m) = -\frac{2s}{\beta} \int \frac{d^3k}{(2\pi)^3} \ln\left(1 + e^{-\beta\varepsilon}\right) \tag{9.19}$$

with energy of particle $\varepsilon = \sqrt{\vec{k}^2 + m^2}$.

The factor $2s = 4$ reflects the existence of the four degrees of freedom present in the fermion field: particles and antiparticles, spin up and spin down.

The following terms are geometrical corrections of the Riemann space time structure with respect to the Euclidean one with temperature coefficients in the form

$$f_j(\beta m) = -\frac{2s}{\beta} \int \frac{d^3k}{(2\pi)^3} \left(-\frac{\partial}{\partial m^2} \right)^j \ln \left(1 + e^{-\beta \varepsilon} \right). \tag{9.20}$$

The method developed above does not allow us to compute the density of the grand thermodynamical potential. Therefore, in the following calculations we will use the local momentum space formalism as the most convenient for the construction of local thermodynamics.

Chapter 10

THERMODYNAMICS OF GAUGE FIELDS

In this chapter we will apply the formalism of gauge fields in curved space-time developed in chapter V to study the properties of thermal photon gas in an external gravitational field.

10.1 The Green's function of photons

As we know already from chapter IV the total Lagrangian for a vector field in Minkowski space-time is the sum of three contributions:

$$L_{tot} = L_m + L_f + L_{gh}, \tag{10.1}$$

where

$$L_m = -(1/4)F_{\mu\nu}F^{\mu\nu}, \tag{10.2}$$

$$L_f = -(1/2\alpha)(\partial_\mu A^\mu)^2 \tag{10.3}$$

and

$$L_{gh} = g^{\mu\nu}(\partial_\mu c)(\partial_\nu c^*) \tag{10.4}$$

It can be extended to curved space-time with the transformation

$$A_\alpha = h_\alpha{}^\mu A_\mu, \quad \partial_\alpha \to h_\alpha{}^\mu \nabla_\mu \tag{10.5}$$

where $\nabla_\mu = \partial_\mu + \Gamma_\mu$ and the connection Γ_μ is defined by equation

$$\Gamma_\mu = (1/2)\Sigma_{\alpha\beta} h^{\alpha\nu}(x)\left[\frac{\partial}{\partial x^\mu} h^\beta{}_n u(x)\right] \tag{10.6}$$

where the matrices $\Sigma_{\alpha\beta}$ are (5.7).

The strength tensor of electromagnetic field is

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu \tag{10.7}$$

The variation of the action

$$S = \int d^4x \sqrt{g}\,(L + L_f + L_{gh}) \tag{10.8}$$

gives the equation for the vector field:

$$\nabla_\nu F^{\mu\nu} + (1/\alpha)\nabla^\mu(\nabla_\nu A^\nu) = 0 \tag{10.9}$$

In the Feynman gauge ($\alpha = 1$) this equation has the form:

$$\nabla_\nu \nabla^\mu A^\nu - \nabla_\nu \nabla^\nu A^\mu + \nabla^\mu \nabla_\nu A^\nu = 0 \tag{10.10}$$

and with the definition of the Riemann tensor (5.17) it is

$$\nabla_\nu \nabla^\nu A^\mu - R^\mu{}_\nu A^\nu = 0 \tag{10.11}$$

Based on this equation we write the equation for the Green's function

$$\nabla_\nu \nabla^\nu D^\mu{}_\tau(x, x^{'}) - R^\mu{}_\nu D^\nu{}_\tau(x, x^{'}) = -g^{-1/2}(x)\delta(x - x^{'})\delta^\mu{}_\tau \tag{10.12}$$

We will be interested in calculations of the Green's function in the limit ($x \to x^{'}$) to find the effective action.

We will rewrite the equation (10.12) in Riemann normal coordinates with origin at the point $x^{'}$. For covenience one may define the Green's function $\bar{D}^\mu_\tau(x, x^{'})$ as

$$D^{\mu}{}_{\tau}(x,x^{'}) = g^{-1/4}(x)\bar{D}^{\mu}_{\tau}(x,x^{'})g^{-1/4}(x^{'})$$

$$= g^{-1/4}(x)\bar{D}^{\mu}_{\tau}(x,x^{'}) \qquad (10.13)$$

where we used the fact $g(x^{'}) = 1$

For our calculations we will use the Christoffel symbols which in the Riemann normal coordinates are

$$\Gamma^{\sigma}{}_{\mu\nu} = -(1/3)(R^{\sigma}{}_{\alpha\beta\gamma} + R^{\sigma}{}_{\beta\alpha\gamma})y^{\gamma} \qquad (10.14)$$

where y^{γ} represents the coordinates of the point x and the Riemann tensor is evaluated at $x^{'}$. The expansion of equation (10.12) to the second derivative of the metric gives

$$\eta^{\alpha\beta}\partial_{\alpha}\partial_{\beta}\bar{D}^{\mu}_{\tau}(y) + (1/6)R\bar{D}^{\mu}_{\tau}(y)$$

$$-(4/3)R^{\mu}_{\nu}\bar{D}^{\nu}_{\tau}(y) - (1/3)R^{\lambda}_{\nu}y^{\nu}\partial_{\lambda}\bar{D}^{\mu}_{\tau}(y)$$

$$+(1/3)R^{\alpha}{}_{\gamma}{}^{\beta}{}_{\delta}y^{\gamma}y^{\delta}\partial_{\alpha}\partial_{\beta}\bar{D}^{\mu}_{\tau}(y) - (2/3)R^{\mu}{}_{\gamma}{}^{\alpha}{}_{\delta}y^{\delta}\partial_{\alpha}\bar{D}^{\gamma}_{\tau}(y)$$

$$+(2/3)R^{\mu\alpha}{}_{\lambda\gamma}y^{\gamma}\partial_{\alpha}\bar{D}^{\lambda}_{\tau}(y) = -\delta(y)\delta^{\mu}_{\tau} \qquad (10.15)$$

where $\partial_{\alpha} = \partial/\partial y^{\alpha}$.

The momentum space approximation is defined by introducing the quantity $D^{\mu}_{\tau}(k)$ defined as

$$\bar{D}^{\mu}_{\tau}(x,x^{'}) = \bar{D}^{\mu}_{\tau}(x^{'},y) = \int \frac{d^n k}{(2\pi)^n} \bar{D}^{\mu}_{\tau}(k) \exp[iky] \tag{10.16}$$

This quantity is assumed to have the expansion

$$\bar{D}^{\mu}_{\tau}(k) = \bar{D}^{\mu}_{0,\tau}(k) + \bar{D}^{\mu}_{1,\tau}(k) + \bar{D}^{\mu}_{2,\tau}(k) + ... \tag{10.17}$$

where $\bar{D}^{\mu}_{i,\tau}(k)$ have a geometric coefficients involving i derivatives of the metric. On dimensional grounds $\bar{D}^{\mu}_{i,\tau}(k)$ must be of order $k^{-(2+i)}$ so the equation (10.15) is an asymptotic expansion in large k.

As we can see from (10.15) the first term in (10.17) is

$$\bar{D}^{\mu}_{0,\tau}(k) = \delta^{\mu}_{\tau}/k^2, \tag{10.18}$$

The second one is

$$\bar{D}^{\mu}_{1,\tau}(k) = 0, \tag{10.19}$$

The third one is computed from

$$\eta^{\alpha\beta}\partial_\alpha\partial_\beta\bar{D}^{\mu}_{2,\tau}(y) + (1/6)R\bar{D}^{\mu}_{0,\tau}(y)$$

$$-(4/3)R^{\mu}_{\nu}\bar{D}^{\nu}_{0,\tau}(y) - (1/3){R_{\nu}}^{\lambda}y^{\nu}\partial_\lambda\bar{D}^{\mu}_{0,\tau}(y)$$

$$+ (1/3)R^{\alpha}{}_{\gamma}{}^{\beta}{}_{\delta}y^{\gamma}y^{\delta}\partial_{\alpha}\partial_{\beta}\bar{D}^{\mu}_{0,\tau}(y) - (2/3)R^{\mu}{}_{\gamma}{}^{\alpha}{}_{\delta}y^{\delta}\partial_{\alpha}\bar{D}^{\gamma}_{0,\tau}(y)$$

$$+ (2/3)R^{\mu\alpha}{}_{\lambda\gamma}y^{\gamma}\partial_{\alpha}\bar{D}^{\lambda}_{0,\tau}(y) + ... = 0 \tag{10.20}$$

To simplify calculations note that $y^{\alpha} \to (-i\partial/\partial k^{\alpha})$, and integrate by parts to find that

$$\bar{D}^{\mu}_{2,\tau}(y) = \int \frac{d^n k}{(2\pi)^n}\bar{D}^{\mu}_{2,\tau}(k)\exp[iky] = \int \frac{d^n k}{(2\pi)^n}\exp[iky]\times$$

$$\times \left[\{(1/6)R\delta^{\mu}_{\tau} - (2/3)R^{\mu}_{\nu}\delta^{\nu}_{\tau}\}/k^4 - (4/3)R^{\mu\beta}{}_{\nu\gamma}k^{\gamma}k_{\beta}\delta^{\nu}_{\tau}/k^6\right] \tag{10.21}$$

The final expression for the photon propagator will be then:

$$D^{\mu}_{\tau}(y) = g^{-1/4}(y)\int \frac{d^n k}{(2\pi)^n}\exp[iky]\times$$

$$\times \left[\delta^{\mu}_{\tau}/k^2 + \{(1/6)R\delta^{\mu}_{\tau} - (2/3)R^{\mu}_{\nu}\delta^{\nu}_{\tau}\}/k^4 - (4/3)R^{\mu\beta}{}_{\nu\gamma}k^{\gamma}k_{\beta}\delta^{\nu}_{\tau}/k^6\right] \tag{10.22}$$

The Green's function of the ghost fields obeys the same equation as a scalar field, therefore one can use this equation for our further calculations.

10.2 The thermodynamic potential of a photon gas

As in the case of flat space-time, one must carry out the calculations of the free energy of a photon gas together with the ghost contributions.

The generating functional of an abelian vector field in curved space-time is written as

$$Z[J_\mu, \eta, \bar{\eta}] = \int DA_\mu D\bar{\eta} D\eta$$

$$\times \exp\left[\int d^4x\sqrt{g}\left\{A^\mu(D_{\mu\nu})^{-1}A^\nu + \bar{c}D^{-1}c + j_\mu A^\mu + \bar{\eta}c + \bar{c}\eta\right\}\right] \qquad (10.23)$$

Integration over the fields with zero sources leads to the following result for the logarithm of the generating functional

$$\ln Z[0] = \ln Det(G) - (1/2)\ln Det(G\mu\nu) \qquad (10.24)$$

where the Green's functions are defined by (10.22) and (6.34)[1]

From this equation and from the definition of the free energy directly follows the expression for density of free energy

$$f_{ph}(\beta, R) = (-i/2)\lim_{m\to 0}\int_{m^2}^{\infty} dm^2\left\{2\mathrm{tr}G(\beta, x - x^{'}) - \mathrm{tr}G_{\mu\nu}(\beta, x - x^{'})\right\} \qquad (10.25)$$

The final result may be found after putting a mass parameter m^2 into the expressions for the propagators of the photon and ghost fields and, after calculation of (10.25) setting the mass equal to zero.

[1]To find the Feynman propagator of the ghost field from the boson propagator we have to put the mass of boson $m = 0$ after calculation.

After making this procedure we will have

$$f_{ph}(\beta, R) = \sum_j g_j(R) \left(-\frac{\partial}{\partial m^2}\right)^j \mathrm{tr}\ln(\omega_n^2 + \epsilon^2) \tag{10.26}$$

where the symbol tr means

$$\mathrm{tr} = \sum_{n=-\infty}^{\infty} \int \frac{d^3k}{(2\pi)^3}$$

and $g_j(R)$ are geometric coefficients.

After summation in (10.26) we will have the final result in the form

$$f_{ph}(\beta, R) = \int \frac{d^3k}{(2\pi)^3} \{(2/\beta)\ln(1 - \exp[\beta\epsilon])$$

$$-(1/6)(R - 2R^\mu_\nu \delta^\nu_\mu)[\epsilon(\exp[\beta\epsilon] - 1)]^{-1}\} \tag{10.27}$$

This expression is the density of the Helmholtz free energy of a photon gas in an external gravitational field.

10.3 Internal energy and heat capacity of a photon gas

Thermodynamic properties of the photon gas have been studied very well in flat space-time [Landau & Lifshitz 1959, Huang 1963]. The formalism developed in this section allows us to get the properties of photon gas in curved space-time.

The density of energy of a photon gas is

$$u = \frac{\partial}{\partial\beta}(\beta f) = \int \frac{d^3k}{(2\pi)^3}\left\{\frac{2\epsilon}{\exp[\beta\epsilon]-1}\right.$$

$$\left.-(1/6)(R - 2R^{\mu}_{\nu}\delta^{\nu}_{\mu})\frac{\partial}{\partial\beta}\int \frac{d^3k}{(2\pi)^3}\frac{\beta}{\epsilon(\exp[\beta\epsilon]-1)}\right\} \tag{10.28}$$

The first term in (10.28) corresponds to the results of statistical mechanics, and the second one is the curved space-time correction to the energy of photon gas. Integrating over the momentum

$$\int \frac{d^3k}{(2\pi)^3}\frac{1}{\epsilon(\exp[\beta\epsilon]-1)} = \frac{1}{12\beta} \tag{10.29}$$

we get the following approximation to $O(R^2)$

$$u(R) = \sigma T^4 + (1/72)(R - 2R^{\mu}_{\nu}\delta^{\nu}_{\mu})T^2 + ... \tag{10.30}$$

where $\sigma = (\pi^2/15)$.

The heat capacity will be

$$c(R) = \frac{\partial}{\partial T}u(R) = 4\sigma T^3 + (1/36)(R - 2R^{\mu}_{\nu}\delta^{\nu}_{\mu})T^2 + ... \tag{10.31}$$

As was shown in [Altaie & Dowker 1978] the effects of curvature for Einstein space lead to the results for the Planck black body expression which are equivalent to the results (10.30).

Chapter 11

RENORMALIZATIONS IN

LOCAL STATISTICAL MECHANICS

11.1 Divergences of finite temperature field models

As one can show by calculations, the contributions $G_{SD}(x,x)$ in the expressions (7.16) and (9.8) are divergent [Birrell & Davies 1982]. To have a clear picture of the model under consideration we need to eliminate these divergent contributions. The direct way to eliminate the divergences is to combine the gravitational Lagrangian and the divergent parts of the matter Lagrangian.

1) Renormalization of $a\ bose\ field$.

As was shown in (7.21) the effective Lagrangian for gravitational field can be written as

$$\tilde{L}_g = L_g(x) + (-i/2) \int_{m^2}^{\infty} \mathrm{tr} G_{SD}(x, x^{'}) \tag{11.1}$$

Inserting (6.48) into the last term of the expression (11.1), we find

$$(-i/2) \int_{m^2}^{\infty} \mathrm{tr} G_{SD}(x, x^{'})$$

$$= \frac{1}{2(4\pi)^2} \sum_{j=0}^{\infty} \gamma_j(x) \int_0^{\infty} ids(is)^{j-3} \exp(-ism^2) \tag{11.2}$$

We will use the procedure of dimensional regularization ['t Hooft & Veltman 1972] to select divergent terms of the expression (11.2). It gives

$$(-i/2) \int_{m^2}^{\infty} \mathrm{tr} G_{SD}(x, x^{'})$$

$$= \frac{1}{2}(4\pi)^{-n/2} \sum_{j=0}^{\infty} \gamma_j(x) \int ids(is)^{(j-1-n/2)} \exp(-ism^2)$$

$$= \frac{1}{2}(4\pi)^{-n/2} \left(\frac{m}{M}\right)^{n-4} \sum_{j=0}^{\infty} \gamma_j(x) m^{4-2j} \Gamma\left(j - n/2\right) \tag{11.3}$$

where n is a dimension of the space-time and M is an arbitrary mass scale. This parameter is introduced to preserve the dimension of the Lagrangian to $[L]^{-4}$ for the dimensions $n \neq 4$.

The functions $\Gamma(z)$ have the poles at $n = 4$:

$$\Gamma\left(-\frac{n}{2}\right) = \frac{4}{n(n-2)}\left(\frac{2}{4-n} - \gamma\right) + O(n-4)$$

$$\Gamma\left(1-\frac{n}{2}\right) = \frac{2}{2-n}\left(\frac{2}{4-n} - \gamma\right) + O(n-4)$$

$$\Gamma\left(2-\frac{n}{2}\right) = \frac{2}{4-n} - \gamma + O(n-4) \tag{11.4}$$

Therefore the first three terms of the expression (11.3) are divergent.

Selecting divergent parts and using the logarithmic expression for

$$\left(\frac{m}{M}\right)^{n-4} = 1 + \frac{1}{2}\ln\frac{m^2}{M^2} + O((n-4)^2) \tag{11.5}$$

write the divergent contributions in the effective Lagrangian $\tilde{L}_g$ in the form

$$(-i/2)\int_{m^2}^{\infty} \mathrm{tr}G_{SD}(x, x^{'})$$

$$= -(4\pi)^{-n/2}\left\{\frac{1}{n-4} + \frac{1}{2}\left[j + \ln\left(\frac{m^2}{M^2}\right)\right]\right\} \times$$

$$\times\left[\frac{4m^2}{n(n-4)} - \frac{2}{n-2}m^2\gamma_1(x^{'}) + \gamma_2(x^{'}) + ...\right] \tag{11.6}$$

Combining (11.6) with the gravitational Lagrangian (7.3) we can redetermine the constants of the gravitational Lagrangian (11.1) as

$$\frac{1}{8\pi G_R}\Lambda_R = \frac{1}{8\pi G_0}\Lambda_0 - \frac{1}{(4\pi)^2}\frac{1}{(n-4)}\frac{m^4}{2} \tag{11.7}$$

$$\frac{1}{16\pi G_R} = \frac{1}{16\pi G_0} + \frac{1}{(4\pi)^2}\frac{1}{n-4}m^2\left(\frac{1}{6} - \xi\right)$$

$$\alpha_R = \alpha_0 - \frac{1}{(4\pi)^2}\frac{1}{n-4}\left(\frac{1}{6} - \xi\right)^2$$

$$\beta_R = \beta_0 + \frac{1}{180}\frac{1}{(4\pi)^2}\frac{1}{n-4}$$

$$\gamma_R = \gamma_0 - \frac{1}{180}\frac{1}{(4\pi)^2}\frac{1}{n-4} \tag{11.8}$$

where G_0, Λ_0, α_0, β_0, γ_0 are bare constants and G_R, Λ_R, α_R, β_R, γ_R are the physical (finite) constants.

Since all divergences of the matter field can be included in $\tilde{L}_g$, only the finite temperature contributions observed in (7.20) remain.

2) Renormalization of *a fermi field*.

As follows from (9.14) the effective Lagrangian of the gravitational field is

$$\tilde{L}_g = L_g - \frac{1}{2}(4\pi)^{-n/2}\sum_{j=0}^{\infty}\mathrm{tr}\hat{\alpha}_j(R)\times$$

$$\times\int_0^{\infty} ids(is)^{j-n/2-1}\exp(-ism^2) \tag{11.9}$$

After dimensional regularization procedure (as in scalar field case), we get

$$\frac{1}{8\pi G_R}\Lambda_R = \frac{1}{8\pi G_0}\Lambda_0 + \frac{1}{(4\pi)^2}\frac{2m^4}{n-4}$$

$$\frac{1}{16\pi G_R} = \frac{1}{16\pi G_0} - \frac{1}{(4\pi)^2}\frac{1}{(n-4)}\frac{m^2}{6}$$

$$\alpha_R = \alpha_0 + \frac{1}{(4\pi)^2}\frac{1}{(n-4)}\frac{1}{144}$$

$$\beta_R = \beta_0 - \frac{1}{(4\pi)^2}\frac{1}{(n-4)}\frac{1}{90}$$

$$\gamma_R = \gamma_0 - \frac{1}{(4\pi)^2}\frac{1}{(n-4)}\frac{7}{720} \qquad (11.10)$$

Thus, the Lagrangian (9.13) does not include the divergencies in the finite temperature loop approximation over the fermi fields.

The method of dimensional regularization developed here is not the only one for applications to field models in background curved space-time. The ζ-function method [Hawking 1977], [Hurt 1983] and the method of covariant geodesic point separation also lead to a solution of the problem of regularization of effective Lagrangians of matter fields and calculation of Energy-momentum tensor anomalies [Christensen 1978].

Chapter 12

LOCAL QUANTUM STATISTICS AND

THERMODYNAMICS OF BOSE GAS

In this chapter we will continue to develop the methods of local statistical thermodynamics [Kulikov & Pronin 1995] with application to the ideal bose gas in curved space-time. In chapter VII working with the Schwinger-DeWitt proper time formalism we found that the density of Helmholtz free energy in curved space-time may be found in the form of a series (7.25) including the geometric structure of space-time. However, as in the previous section, working with this formalism we could have certain difficulties with the introduction of the chemical potential and the computing of the density of the grand thermodynamical potential.

To avoid such problems which will appear on the way of construction of local thermodynamics with the Schwinger DeWitt formalism, we turn to the local momentum space method [Bunch & Parker 1979], [Panangaden 1981] in quantum field theory in an arbitrary curved space time and the imaginary time formalism to introduce the temperature [Dolan & Jackiw 1974], [Weinberg 1974].

For the construction of the local quantum statistics and thermodynamics of a bose gas, we will use the connection between the partition function of ideal quantum systems and finite temperature Green's functions, which may be found by the local momentum space method.

12.1 Density of grand thermodynamical potential

In this section we will analyze the scalar field model with a conserved charge. The Lagrangian of the model is

$$S_m = -(1/2) \int d^4x \sqrt{g(x)} \Phi^*(x)(-\Box_x + m^2 + \xi R)\Phi(x) \tag{12.1}$$

where $\Phi = (\phi_1, \phi_2)$ is a dublet of the real fields.

The action written in terms of real fields will be

$$S_m = -(1/2) \int d^4x \sqrt{g(x)} \phi^a(x)(-\Box_x + m^2 + \xi R)\phi^a(x) \tag{12.2}$$

The total action of the system "matter + gravity" is

$$S_{tot} = S_g + S_m \tag{12.3}$$

Now we can write the effective action at finite temperature in analogy with (7.20) as

$$L_{eff}(\beta) = \tilde{L}_g - \omega(\beta, \mu, R) \tag{12.4}$$

where $\tilde{L}_g$ is (7.21) and $\omega(\beta, \mu, R)$ is the density of grand thermodynamic potential.

The result (12.4) may be obtained with the momentum space representation for the Green's function of a boson (6.34) [Kulikov & Pronin 1987].

In the momentum space representation, the expression for L_{eff} is split into two parts

$$L_{eff} = -(i/2) \int\limits_{m^2}^{\infty} dm^2 \mathrm{tr} G(x, x^{'}) - \omega(\beta, \mu, R) \tag{12.5}$$

The potential $\omega(\beta, R)$ is

$$\omega(\beta, R) = (-1/2)\mathrm{tr} \int\limits_{m^2}^{\infty} dm^2 \sum_{j=0}^{2} \sum_{n=0}^{\infty} \gamma_j(x, x^{'}) \left(-\frac{\partial}{\partial m^2}\right)^j \times$$

$$\times \int \frac{d^3k}{(2\pi)^3} ({\omega_n}^2 + \epsilon^2)^{-1}$$

$$= (1/2) \sum_{j=0}^{2} \gamma_j(x^{'}) \left(-\frac{\partial}{\partial m^2}\right)^j \mathrm{tr} \ln({\omega_n}^2 + \epsilon^2) \tag{12.6}$$

and coincides with the Helmholtz free energy (7.22) and $\omega_n = 2\pi n/\beta$.

The symbol tr in (12.6) is determined as

$$\mathrm{tr}(...) = \sum_{n\neq 0} \int \frac{d^3k}{(2\pi)^3} ...$$

For introducing the chemical potential, we will change the Matsubara frequencies $\omega_n \to \omega_n + \mu$ and thermodynamic potential will be $\omega(\beta, \mu, R)$.

Since both positive and negative frequencies are summed, we will have

$$\mathrm{tr}\ln({\omega_n}^2 + \epsilon^2) \to \mathrm{tr}\ln[(\omega_n + \mu)^2 + \epsilon^2]$$

$$= \mathrm{tr}\left\{\ln\left[{\omega_n}^2 + (\epsilon - \mu)^2\right] + \ln\left[{\omega_n}^2 + (\epsilon + \mu)^2\right]\right\} \qquad (12.7)$$

After summation in (12.7) we get

$$\omega(\beta, \mu, R) = \omega_-(\beta, \mu, R) + \omega_+(\beta, \mu, R) \qquad (12.8)$$

where

$$\omega_-(\beta, \mu, R) = (1/\beta)\sum_{j=0}^{2} \gamma_j(x^{'}) \left(-\frac{\partial}{\partial m^2}\right)^j \ln(1 - \exp[-\beta(\epsilon - \mu)]) \qquad (12.9)$$

and

$$\omega_+(\beta, \mu, R) = (1/\beta)\sum_{j=0}^{2} \gamma_j(x^{'}) \left(-\frac{\partial}{\partial m^2}\right)^j \ln(1 - z\exp[-\beta(\epsilon + \mu)]) \qquad (12.10)$$

So, with accordance to (12.8) the density of grand thermodynamic potential is the series

$$\omega(\beta, \mu, R) = \sum_{j=0}^{2} \gamma_j^{'}(x^{'}) b_j(\beta m, z) \tag{12.11}$$

where

$$b_0(\beta m, z) = (1/\beta) \ln(1 - z \exp(-\beta\epsilon));$$

$$b_j(\beta m, z) = \left(-\frac{\partial}{\partial m^2}\right)^j b_0(\beta m, z) \tag{12.12}$$

and the fugacity is $z = \exp(\beta\mu)$. The geometrical coefficients $\gamma_j^{'}(R)$ of the equation (12.12) have the form (6.47). Renormalizations in the total Lagrangian (12.4) are the same as in chapter XI.

12.2 Statistics and thermodynamics of bose gas

We find the bose distribution function as the derivative of the grand thermodynamical potential

$$n_{\vec{k}} = -\frac{\partial \omega_{\vec{k}}(\beta, \mu, R)}{\partial \mu}$$

For occupation numbers with momentum $\vec{k}$ we get

$$n_{\vec{k}} = \frac{1}{(z^{-1}e^{\beta\varepsilon_{\vec{k}}} - 1)}B(\beta, R), \tag{12.13}$$

where the function $B(\beta, R)$ is described by the formula

$$B(\beta, R) = \ 1 + \gamma_1(R)\frac{\beta}{2\varepsilon_{\vec{k}}}\left[1 - (1 - ze^{-\beta\varepsilon_{\vec{k}}})^{-1}\right] + \quad \ldots \tag{12.14}$$

The function $B(\beta, R)$ depends on the curvature, temperature and energy of the boson.

Studying the thermodynamical properties of Bose gases we will start with the equation

$$\omega(\beta, \mu, R) = -(1/\beta)\sum_{j=0}^{2}\gamma_j(x^{'})\left(-\frac{\partial}{\partial m^2}\right)^j \ln(1 - z\exp[-\beta\epsilon]) \tag{12.15}$$

In the non-relativistic limit for particle energy $\epsilon = \vec{k}^2/2m$ we get from (12.15) the equation

$$\omega(\beta, \mu, R) = \sum_{j=0}^{2}\gamma_j(x^{'})g_{5/2}(z)\left(-\frac{\partial}{\partial m^2}\right)^j \lambda^{-3} \tag{12.16}$$

where $\lambda = (2\pi/mT)^{1/2}$ is a wavelength of the particle, and the function $g_{5/2}(z)$ has the following form

$$g_{5/2}(z) = \sum_{l=1}^{\infty}\frac{z^l}{l^{5/2}}$$

$$= -(4/\sqrt{\pi})\int dx x^2 \ln\left(1 - z\exp(-x^2)\right) \tag{12.17}$$

The average number of particles in a certain momentum state k is obtained as the derivative

$$< n_k >= -\frac{\partial}{\partial \mu}\omega(\beta, \mu, R)$$

$$= \sum_{j=0}^{2} \gamma_j(x^{'}) g_{5/2}(z) \left(-\frac{\partial}{\partial m^2}\right)^j \left(z^{-1}\exp(\beta\epsilon) - 1\right) \tag{12.18}$$

The density of particles is

$$n = \lambda^{-3}\left[1 - \gamma_1(R)(3/4m^2) - \gamma_2(R)(3/16m^4)\right] g_{3/2}(z) + n_0 \tag{12.19}$$

where the new function $g_{3/2}(z)$ is

$$g_{3/2}(z) = z\frac{\partial}{\partial z} g_{5/2}(z)$$

and $n_0 = z/(1-z)$ is the average number of particles with zero momentum. The functions $g_{3/2}(z)$ and $g_{5/2}(z)$ are special cases of a more general class of functions

$$g_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n} \tag{12.20}$$

In a more simple form the equation(12.19) may be written as

$$(n - n_0)\lambda^3 = g_{3/2}(z, R) \tag{12.21}$$

where

$$g_{3/2}(z,R) = \left[1 - \alpha\frac{R}{m^2} + ...\right] g_{3/2}(z) \tag{12.22}$$

is a function which depends on curvature, and α is a numerical parameter.

The equation (12.21) connects four values: fugacity, temperature, density of the particles and curvature.

We can solve it graphically and get the dependence of the (effective) chemical potential on curvature, temperature and density. The function $g_{3/2}(z,R)$ varies with the curvature R as shown in Fig. I-1. The graphical solution of the equation (12.21) is presented in Fig. I-2. In this Figure X-projection of the point of intersection of the curve $g_{3/2}(z,R) + \lambda^3 n_0$ and of the line $\lambda^3 n$ shows the dependence of the fugacity on curvature. As follows from the graphical picture, for positive curvature $R < 0$ the fugacity $z(R) > z_0$ (or $\mu(R) > \mu_0$), for $R > 0$ we have $z(R) < z_0$ (or $\mu(R) > \mu_0$). The fugacity z_0 (μ_0-chemical potential) corresponds to statistics in Euclidean space. The behaviour of the effective chemical potential is shown in Fig I-3.

12.3 Bose-Einstein condensation

At low temperature there is a significal number of particles in the ground state, which can be expressed by the equation

$$n_0 = n - \lambda^{-3} g_{3/2}(z,R) \tag{12.23}$$

With the rising of the temperature the average number of particles n_0 with zero momentum is lowered and for temperatures $T > T_c$ it becomes zero. The temperature T_c is the critical temperature of a Bose condensation. The critical temperature may be found from the equation with ($z = 1$ and $n_0 = 0$)

$$n{\lambda_c}^3 = g_{3/2}(1, R) \tag{12.24}$$

The solution of this equation is [Kulikov & Pronin 1993]

$$T_c(R) = T_0 \left(1 + \frac{\gamma_1'(R)}{2m^2} + ...\right) \tag{12.25}$$

The temperature

$$T_0 = T_c(R = 0) = \frac{2\pi}{m} \left[\frac{n}{\zeta(3/2)}\right]^{2/3} \tag{12.26}$$

where $\zeta(3/2) = 2.612...$ is the Riemann zeta function, n is the density of bosons and m is the mass of boson is degeneracy temperature (condensation temperature) in "flat" space without gravity.

The ratio

$$\frac{T_c(R)}{T_0} = 1 + \frac{R}{12m^2} + ..., \tag{12.27}$$

is not one, but depends on the curvature R of space-time.

As we can see, $T_c(R) > T_0$, for $R > 0$ and $T_c(R) < T_0$ for $R < 0$. The correction

$$\frac{\delta T_c(R)}{T_0} = \frac{R}{12m^2} + O\left((R/m^2)^2\right) \tag{12.28}$$

is small, therefore effects of curvature will be essential for quantum systems in strong gravitational fields.

Chapter 13

LOCAL STATISTICS AND

THERMODYNAMICS OF FERMI GAS

The results of the previous chapter XII show the way to construct thermodynamic potentials of quantum systems with a variable number of particles. In this chapter we will develop this formalism for fermi systems.

13.1 Grand thermodynamical potential and low temperature properties of Fermi gases

An equivalent to the Schwinger-DeWitt representation, the momentum space representation of the bi-spinor $G_F(x, x^{'})$ has the following form (8.14):

$$G_F(x, x^{'}) = G_F(x, y) = g^{-\frac{1}{4}}(y) \sum_{j=0}^{2} \hat{\alpha}_j(x, y) \left(-\frac{\partial}{\partial m^2}\right)^j G_0(y) \tag{13.1}$$

where

$$G_0(y) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{iky}}{k^2+m^2} \tag{13.2}$$

To introduce temperature, we will extend the time y^0 coordinate of the tangential space $\{y^\mu\}$ to the imaginary interval $[0, -i\beta]$ and will consider fermionic field to be antiperiodic on that interval. Then, in imaginary time formalism, we can write

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2+m^2} \xrightarrow{\beta} \frac{i}{\beta} \int \frac{d^3k}{(2\pi)^3} \sum_{n=-\infty}^{\infty} \frac{1}{\omega_n^2+\varepsilon^2} \tag{13.3}$$

where $\omega_n = \pi T(2n+1), \quad n = 0, \pm 1, \pm 2, \quad \ldots$ are Matsubara frequencies. To take into consideration the chemical potential we will shift the frequencies by $\omega_n \to \omega_n + \mu$ [Morley 1978, Kapusta 1979]

Making the summation in (13.3) we will find $G_0(y)$ in the limit of coincidence $x = x'$ in the form

$$\lim_{x\to x'} G_0(y) \stackrel{\beta}{=} \frac{i}{\beta} \int \frac{d^3k}{(2\pi)^3} \sum_{n=-\infty}^{\infty} \frac{1}{-(\omega_n+\mu)^2+\varepsilon^2}$$

$$= \frac{i}{\beta} \int \frac{d^3k}{(2\pi)^3} \left\{ \frac{1}{2\varepsilon} \sum_{n=-\infty}^{\infty} \left(\frac{\varepsilon-\mu}{(\varepsilon-\mu)^2-\omega_n^2} + \frac{\varepsilon+\mu}{(\varepsilon+\mu)^2-\omega_n^2} \right) \right\}$$

$$= \int \frac{d^3k}{(2\pi)^3} \left\{ \frac{i}{2\varepsilon} \left(\frac{1}{2} - \frac{1}{\exp\beta(\varepsilon-\mu)+1} \right) + \frac{i}{2\varepsilon} \left(\frac{1}{2} - \frac{1}{\exp\beta(\varepsilon+\mu)+1} \right) \right\}$$

$$= \lim_{x\to x'} \left[G_\beta^+(y) + G_\beta^-(y) \right] \tag{13.4}$$

This equation describes the temperature contributions for particles (μ) and antiparticles ($-\mu$) separately.

As a result one can find the expression for the finite temperature contribution to the Green's function for a fermion with non-zero chemical potential

$$G_\beta(x,\mu) = \int \frac{d^3k}{(2\pi)^3} \sum_{j=0}^{2} \hat{\alpha}_j(R) \left(\frac{\partial}{\partial m^2}\right)^j \left(1 + ze^{-\beta\varepsilon}\right)^{-1} \tag{13.5}$$

Then the density of grand thermodynamical potential for fermions may be written as

$$\omega(\beta,\mu,R) = -\frac{i}{2}\mathrm{tr}\int_{m^2}^{\infty} dm^2 G_\beta(x,\mu) = \sum_{j=0}^{2} \alpha_j(R) f_j(\beta m; z), \tag{13.6}$$

where

$$f_0(\beta m; z) = -\frac{s}{\beta}\int \frac{d^3k}{(2\pi)^3} \ln(1 + ze^{-\beta\varepsilon}) \tag{13.7}$$

and

$$f_j(\beta m; z) = \left(-\frac{\partial}{\partial m^2}\right)^j f_0(\beta m; z), \tag{13.8}$$

where $z = e^{\beta\mu}$ is the fugacity and factor $s = 2$ (spin up, down).

The coincidence of the finite temperature Schwinger-DeWitt and momentum space methods for calculation of the densities of themodynamical potentials is obvious for $\mu = 0$.

Using the equations for thermodynamic potentials one can obtain some interesting properties of an ideal fermi gas in an external gravitational field :

1) The Fermi distribution function of the gas in the gravitational field may be found from the expression

$$n_{\vec{k}} = -\frac{\partial \omega_{\vec{k}}(\beta, \mu, R)}{\partial \mu}$$

for occupation numbers with momentum $\vec{k}$.

It has the form

$$n_{\vec{k}} = \frac{1}{(z^{-1}e^{\beta\varepsilon_{\vec{k}}} + 1)} F(\beta, R), \tag{13.9}$$

where the function $F(\beta, R)$ is described by the formula

$$F(\beta, R) = \; 1 + \alpha_1(R)\frac{\beta}{2\varepsilon_{\vec{k}}} \left[1 - (1 + ze^{-\beta\varepsilon_{\vec{k}}})^{-1}\right] + \quad \ldots \tag{13.10}$$

and depends on curvature, temperature and energy of the fermion.

2) We can estimate the dependence of the chemical potential on the curvature of space time in non-relativistic approximation.

Let

$$\varepsilon_{\vec{k}} = \frac{\vec{k}^2}{2m} \tag{13.11}$$

then from

$$n = -\frac{\partial \omega(\beta, \mu, R)}{\partial \mu} \tag{13.12}$$

we find the equation

$$\frac{n\lambda^3(T)}{s} = f_{\frac{3}{2}}(z, R), \tag{13.13}$$

where $\lambda = (2\pi/mT)^{1/2}$ is the thermal wave length of the particle, and

$$f_{3/2}(z, R) = f_{3/2}(z) \left[\alpha_0 - \frac{3}{4}\frac{\alpha_1(R)}{m^2} - \frac{3}{16}\frac{\alpha_2(R^2)}{m^4} - ... \right] \tag{13.14}$$

is some function with respect to z, and n is the density of Femi gas.

The function $f_{\frac{3}{2}}(z)$ is

$$f_{3/2}(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^{3/2}} (-1)^{n+1} = \frac{4}{\sqrt{\pi}} \int_0^{\infty} dx \frac{x^2}{z^{-1}\exp(x^2) + 1}. \tag{13.15}$$

The equation (13.13) may be solved with graphical methods. As we can see in Fig. I-4 the fugacity (chemical potential) depends on the curvature R of the space time.

3) The explicit expression for the chemical potential at low temperatures and high densities ($n\lambda^3 >> 1$), where quantum effects are essential, may be found by the calculations of (13.13) with the following representation of the function $f_{3/2}(z)$ for large z [Huang 1963]

$$f_{3/2}(z) = \frac{4}{3\sqrt{\pi}} \left[(\ln z)^{3/2} + \frac{\pi^2}{8} (\ln z)^{-1/2} + \quad \ldots \right] + O(z^{-1}). \tag{13.16}$$

Inserting (13.14) into (13.13) and taking into account only the first term in (13.16) we find the Fermi energy of a gas of fermions in curved space-time

$$\left(\frac{3n\sqrt{\pi}}{4s} \right)^{2/3} \lambda^2 \left[1 + \frac{1}{24} \frac{R}{m^2} + \quad \ldots \right] = \beta \varepsilon_F(R), \tag{13.17}$$

or

$$\varepsilon_F(R) = \varepsilon_F^{(0)} \left[1 + \frac{1}{24} \frac{R}{m^2} + \quad \ldots \right], \tag{13.18}$$

where

$$\varepsilon_F^{(0)} = \left(\frac{6\pi^2 n}{s} \right)^{\frac{2}{3}} \left(\frac{1}{2m} \right) \tag{13.19}$$

is the Fermi energy at the Euclidean space.

Taking into account the second term in (13.16) we get the expression for chemical potential

$$\mu(T, R) = \varepsilon_F(R) \left\{ 1 - \frac{\pi^2}{12} \left(\frac{T}{\varepsilon_F(R)} \right)^2 + \quad \ldots \right\}$$

$$= \varepsilon_F^{(0)} \left\{ 1 + \frac{1}{24} \frac{R}{m^2} - \frac{\pi^2}{12} \left(\frac{T}{\varepsilon_F^{(0)}} \right)^2 + \quad \ldots \right\}$$

or

$$\mu(T,R) = \mu^{(0)}(T) + \frac{1}{24}\frac{R}{m^2}\varepsilon_F^{(0)} + \quad \ldots \tag{13.20}$$

which describes the explicit dependence of the chemical potential of the fermionic gas on temperature T and the curvature R of space-time.

Conclusion

The thermodynamical potentials of quantum bose and fermi gases (as local thermodynamical objects in a curved space-time) were rewritten in terms of the finite temperature Green's functions of bosons and fermions by means of the local momentum space formalism. The phenomenon of Bose condensation was studied and the critical temperature of condensation as a functional of curvature was found (12.25). The non-thermal character of Bose and Fermi distribution functions (12.13) and (13.9) was shown. The dependence of Fermi energy (13.18) and the chemical potential (13.20) of a fermi gas on the curvature of space-time was computed for low temperatures. The dependence of chemical potential of bose gas on the curvature of space-time was analyzed. It was found that the temperature is a local thermodynamical characteristic of thermal systems in external gravitational fields.

PART II

INTERACTING FIELDS AT FINITE TEMPERATURE

Introduction

In Part I of this work we studied ideal quantum systems (systems in which particles don't interact with each other, but only with external fields) at definite temperatures. We described statistical properties and thermodynamical behavior of quantum ensembles of these particles in an arbitrary curved space-time. However can develop a formalism of finite temperature field theory for applications to the systems of interacting particles (bosons or fermions) and study properties of such systems in external gravitational fields.

Part II is devoted to studying thermal interacting quantum systems in gravitational fields. In this part we study renormalizability of the finite temperature self-interacting scalar $\lambda\varphi^4$ model, and consider the phenomenon of non-equality between inertial and gravitational masses of the boson in perturbative regime at finite temperature.

In Part II the following topics are developed: In chapter XIV $\lambda\varphi^4$ model at two-loop perturbative regime is considered. The concepts of renormalizations and all necessary counterterms are described. In chapter XV a complete renormalization procedure for finite temperature model $\lambda\varphi^4$ in two loop approximation of the perturbative scheme is developed. The Green's function of a boson in a heat bath is computed.

In chapter XVI the finite temperature Hamiltonian of a boson is constructed. The phenomenon of non-equality between inertial and gravitational masses of a boson in non-relativistic aproximation at high temperature in the heat bath is described.

Chapter 14

TWO-LOOP RENORMALIZATIONS IN $\lambda\phi^4$ MODEL

We will start with the problem of renormalization procedure for the self-interacting scalar model in two-loop approximation of perturbation theory. In our calculations we will use the method of counterterms.

Let the Lagrangian of the self-interacting $\lambda\varphi^4$ model be

$$L = \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m_B^2\varphi^2 - \frac{\lambda_B}{4!}\varphi^4 \qquad (14.1)$$

We may assume that (14.1) is

$$L = L_0 + L_I \qquad (14.2)$$

where the Lagrangian

$$L_0 = \frac{1}{2}\left(\partial\varphi\right)^2 - \frac{1}{2}m_R^2\varphi^2 \tag{14.3}$$

describes propagation of free particles, and the Lagrangian

$$L_I = -\frac{1}{2}\delta m^2\varphi^2 - \frac{\lambda_B}{4!}\varphi^4 \tag{14.4}$$

describes interaction.

Let the coefficient δm^2 be the difference of the form

$$\delta m^2 = m_B^2 - m_R^2$$

and constant λ_B is expressed as

$$\lambda_B = \mu^{4-n}(\lambda_R + \delta\lambda) \tag{14.5}$$

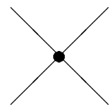
We determine m_B and m_R as bare and renormalizable boson masses and λ_B and λ_R as bare and renormalizable constants of interaction.

It is easy to see from (14.3) and (14.4) that the free propagator of a scalar field is

$$G = i/(p^2 - m_R^2) =$$

and its vertex is

$$-i\lambda_R\mu^{4-n} =$$

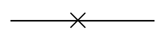


Feynman rules for this model have a standard form [Itzykson & Zuber 1980].

The Feynman diagrams of the counterterms may be found from the Lagrangian of interaction (14.4).

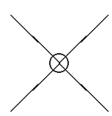
The two point counterterm diagram is

$$-i\delta m^2 =$$

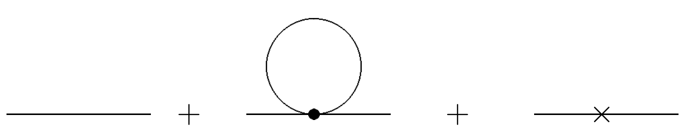


and the vertex counterterm is

$$-i\delta\lambda\mu^{4-n} =$$



Taking into account these counterterms one can construct contributions of the order λ_B to Feynman propagator G

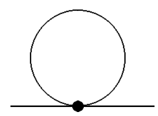


a) b) c)

Fig. II-1 One loop and counterterm contributions to the self energy of the boson.

Propagator Fig. 1a) has a standard form.

The self-energy diagram of the first order to λ_B for Fig. 1-b) may be constructed from the above mentioned graphs in the form

$Fig.1b \quad =$  $=$

$$= \frac{1}{2}\frac{\left(-i\lambda_R\mu^{4-n}\right)}{(2\pi)^n}\int d^nk\frac{i}{k^2-m_R^2}$$

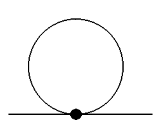
$$= \frac{1}{2}\frac{\left(\lambda_R\mu^{4-n}\right)}{(2\pi)^n}\int d^nk\frac{1}{k^2-m_R^2} \tag{14.6}$$

The factor (1/2) is the symmetry factor and μ is a parameter with dimension of mass. This parameter is used to absorb the dimension of the coupling constant.

Using the expression [Ryder 1985] for the n-dimensional integral

$$\int d^nk\frac{1}{\left(k^2+2kq-m^2\right)^\alpha} = (-1)^\alpha\, i\pi^{\frac{n}{2}}\frac{\Gamma\left(\alpha-\frac{n}{2}\right)}{\Gamma(\alpha)}\frac{1}{\left(q^2+m^2\right)^{\alpha-\frac{n}{2}}} \tag{14.7}$$

we find

$Fig.1b \quad =$  $=$

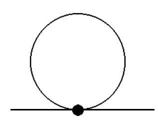
$$= \frac{\lambda_R}{2} \frac{\mu^{4-n}}{(2\pi)^n} \frac{\left(-i\pi^{\frac{n}{2}}\right)}{(m^2)^{1-\frac{n}{2}}} \Gamma\left(1 - \frac{n}{2}\right) =$$

$$= \frac{-i\lambda_R}{2} \frac{m^2}{16\pi^2} \left(\frac{m^2}{4\pi\mu^2}\right)^{\frac{n}{2}-1} \Gamma\left(1 - \frac{n}{2}\right) \qquad (14.8)$$

From the expression for gamma function

$$\Gamma\left(1 - \frac{n}{2}\right) = \frac{2}{n-4} + \gamma - 1 \qquad (14.9)$$

finally get the divergent part of the self-energy diagram

$Fig.1a$ =  =

$$= -\frac{i\lambda_R}{16\pi^2} \frac{m^2}{n-4} + \lambda_R \times finite\ term + O\left(\lambda_R^2\right) \qquad (14.10)$$

The renormalization of the first order of λ_R may be done with the equation

$$i\Gamma^{(2)} = i\left[\frac{p^2 - m_R^2}{i} + (-i\delta m^2) - \frac{i\lambda_R m_R^2}{16\pi^2} \frac{1}{n-4} + ...\right] \qquad (14.11)$$

Let us express term δm^2 in the form of series [Collins 1974]

$$\delta m^2 = m_R^2 \sum_{\nu=1}^{\infty} \sum_{j=\nu}^{\infty} \frac{b_{\nu j} \lambda_R^j}{(n-4)^\nu} =$$

$$= m_R^2 \left\{ \frac{b_{11}\lambda_R}{(n-4)} + \frac{b_{12}\lambda_R^2}{(n-4)} + \frac{b_{22}\lambda_R^2}{(n-4)^2} + ... \right\} \tag{14.12}$$

where the coefficients $b_{\nu j}$ are the numbers.

In order for vertex $\Gamma^{(2)}$ in one loop approximation to be finite

$$i\Gamma^{(2)} = p^2 - m_R^2 + \left(\delta m^2 + \frac{\lambda_R m_R^2}{16\pi^2} \frac{1}{n-4} \right) = finite \tag{14.13}$$

assume

$$\delta m^2 = -\frac{\lambda_R m_R^2}{16\pi^2} \frac{1}{n-4} \tag{14.14}$$

then

$$b_{11} = -\frac{1}{16\pi^2} \tag{14.15}$$

This result completes the one loop calculations.

Now we will find the vertex corrections for $\Gamma^{(4)}$.

There are four graphs of the order λ_R^2:

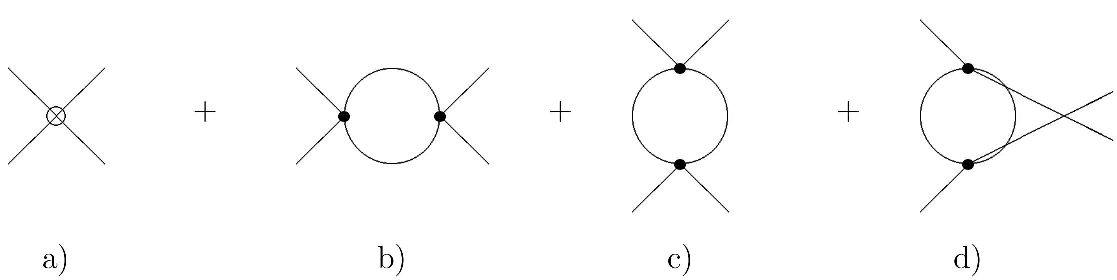
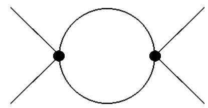


Fig. II-2 Feynman diagrams contributing to the vertex correction $\Gamma^{(4)}$.

The vertex counterterm Fig. 2a) was introduced before.

The loop contribution (Fig. 2b)) of the order λ_R^2 to the vertex function is



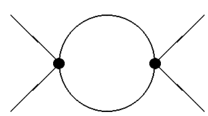
$=$

$$= \frac{1}{2}\frac{(-i\lambda_R\mu^{4-n})^2}{(2\pi)^{2n}}\int d^nk \frac{(i)^2}{(k^2-m^2)\left[(p+k)^2-m^2\right]} =$$

$$= \frac{1}{2}\frac{\lambda_R^2\mu^{8-n}}{(2\pi)^{2n}}\int d^nk \int_0^1 dx \frac{1}{\left[(k^2-m^2)(1-x)+\left[(p+k)^2-m^2\right]x\right]^2} =$$

$$= \frac{1}{2}\frac{\lambda_R^2\mu^{8-n}}{(2\pi)^{2n}}\int_0^1 dx \int d^nk \frac{1}{\left[k^2+2pkx-(m^2-p^2x)\right]^2} \tag{14.16}$$

Taking into account (14.7) we get



$=$

$$= \frac{\lambda_R^2\mu^{8-n}i\pi^{\frac{n}{2}}}{2(2\pi)^{2n}}\Gamma\left(2-\frac{n}{2}\right)\int_0^1 dx \frac{1}{\left[m^2-p^2x(1-x)\right]^{2-\frac{n}{2}}} =$$

$$= -\frac{i\lambda_R^2}{16\pi^2}\frac{1}{n-4} + \lambda_R^2 \times finite \;\; term \tag{14.17}$$

Here we used the representation for $\Gamma(2-n/2)$ in the form

$$\Gamma\left(2-\frac{n}{2}\right)=-\frac{2}{n-4}-\gamma \tag{14.18}$$

Two other loops have the same divergent contributions, therefore $\Gamma^{(4)}$ vertex structure can be expressed in the form

$$\Gamma^{(4)}=-i\lambda_R\mu^{4-n}-\frac{3}{16\pi^2}\frac{\lambda_R^2\mu^{4-n}}{(n-4)}-i\delta\lambda\mu^{4-n}=$$

$$\Gamma^{(4)}=-i\lambda_R\mu^{4-n}-i\left[\frac{3}{16\pi^2}\frac{\lambda_R^2\mu^{4-n}}{(n-4)}+\delta\lambda\right]\mu^{4-n}=finite \tag{14.19}$$

From the equation (14.19) we find, that

$$\delta\lambda=-\frac{3}{16\pi^2}\lambda_R^2 \tag{14.20}$$

Putting

$$\lambda_B=\mu^{4-n}(\lambda_R+\delta\lambda)$$

$$=\mu^{4-n}\left[\lambda_R^2+\sum_{\nu=1}^{\infty}\sum_{j=\nu}^{\infty}\frac{a_{\nu j}\lambda_R^j}{(n-4)^{\nu}}\right] \tag{14.21}$$

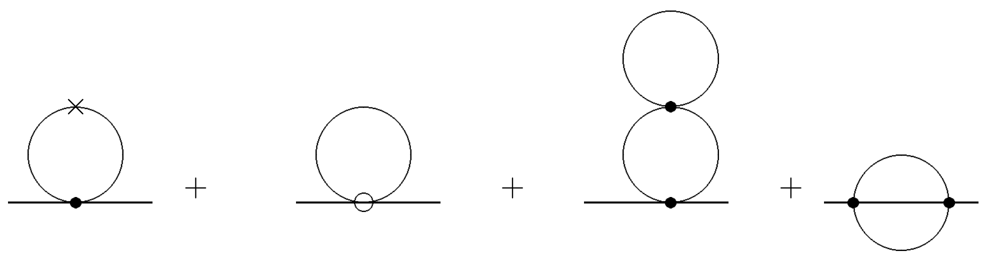
we find coefficient a_{12}:

$$a_{12}=-\frac{3}{16\pi^2} \tag{14.22}$$

Therefore

$$\lambda_B = \mu^{4-n}\left[\lambda_R - \frac{3}{16}\frac{\lambda_R^2}{(n-4)} + O\left(\lambda_R^3\right)\right] \tag{14.23}$$

Further we will consider two-loop contributions to two-point vertex $\Gamma^{(2)}$

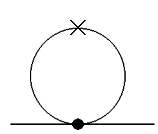


a) b) c) d)

Fig. II-3 Counterterms and the loop contributions of the order λ_R^2 to the self energy.

To find counterterms to the order $O(\lambda_R^3)$, we will make loop calculation of all these contributions.

Diagram 3(a) gives

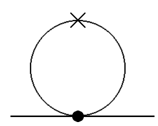
$Fig.3a$ =  =

$$= \frac{1}{2}\frac{\left(-i\lambda_R\mu^{4-n}\right)}{(2\pi)^n}\left(-i\delta m^2\right)\int d^n k \frac{(i)^2}{\left(k^2-m^2\right)^2} =$$

$$= \frac{1}{2}\frac{\lambda_R\mu^{4-n}}{(2\pi)^n}\left(\delta m^2\right)\frac{i\pi^{\frac{n}{2}}\Gamma\left(2-\frac{n}{2}\right)}{\left(m^2\right)^{2-\frac{n}{2}}} =$$

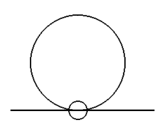
$$= \frac{\lambda_R}{32\pi^2}(\delta m^2)\Gamma\left(2-\frac{n}{2}\right)\left(\frac{4\pi\mu^2}{m^2}\right)^{2-\frac{n}{2}} \tag{14.24}$$

Inserting (14.18) into (14.24) we get

$Fig.3a \quad =$ 

$$= \frac{i\lambda_R}{(16\pi^2)^2}\frac{m_R^2}{(n-4)^2} + \frac{i\gamma\,\lambda_R m_R^2}{2(16\pi^2)^2}\frac{1}{(n-4)} + finite\,terms \tag{14.25}$$

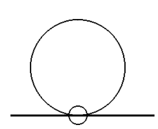
Diagram 3(b) gives

$Fig.3b \quad =$  $=$

$$= \frac{1}{2}\frac{(-i\delta\lambda\mu^{4-n})}{(2\pi)^n}\int d^n k \frac{i}{k^2-m^2} =$$

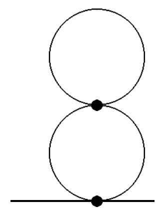
$$= \frac{1}{2}\frac{\delta\lambda\mu^{4-n}}{(2\pi)^n}\left(-i\pi^{\frac{n}{2}}\right)\frac{\Gamma\left(1-\frac{n}{2}\right)}{(m^2)^{1-\frac{n}{2}}} \tag{14.26}$$

Inserting (14.9) into (14.26) we find

$Fig.3b \quad =$  $=$

$$= \frac{3i}{(16\pi^2)^2}\frac{\lambda_R^2}{(n-4)^2} + \frac{3i}{2(16\pi^2)^2}\frac{\lambda_R^2}{(n-4)}(\gamma - 1) \tag{14.27}$$

Diagram 3(c) gives:

$Fig.3c$ =  =

$$= \frac{1}{4}\frac{\left(-i\lambda_R\mu^{4-n}\right)^2}{(2\pi)^{2n}}\int d^n k \int d^n l \frac{(i)^3}{\left(k^2-m^2\right)^2\left(l^2-m^2\right)} =$$

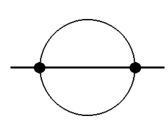
$$= \frac{i}{4}\frac{\lambda_R^2\mu^{8-2n}}{(2\pi)^{2n}}\int d^n k \frac{1}{\left(k^2-m^2\right)^2}\int d^n l \frac{1}{\left(l^2-m^2\right)} =$$

$$= \frac{i}{4}\frac{\lambda_R^2\mu^{8-2n}}{(2\pi)^{2n}}\left\{\left(i\pi^{\frac{n}{2}}\right)\frac{\Gamma\left(2-\frac{n}{2}\right)}{\left(m^2\right)^{2-\frac{n}{2}}}\right\}\left\{\left(-i\pi^{\frac{n}{2}}\right)\frac{\Gamma\left(1-\frac{n}{2}\right)}{\left(m^2\right)^{1-\frac{n}{2}}}\right\} =$$

$$= \frac{i}{4}\frac{\lambda_R^2}{(16\pi^2)^2}\left(\frac{4\pi\mu^2}{m^2}\right)^{4-n}\Gamma\left(2-\frac{n}{2}\right)\Gamma\left(1-\frac{n}{2}\right) =$$

$$= \frac{-i\lambda_R^2}{(16\pi^2)^2}\frac{m_R^2}{(n-4)^2} - \frac{i\lambda_R^2}{(32\pi^2)^2}\frac{m_R^2}{(n-4)}[4\gamma - 2] + finite\ terms \tag{14.28}$$

Diagram 3(d) is

$Fig.3d$ =  =

$$
= \frac{1}{6} \frac{\left(-i\lambda_R \mu^{4-n}\right)^2}{(2\pi)^{2n}} \int d^n k \int d^n l \frac{(i)^3}{\left(k^2 - m^2\right)\left(l^2 - m^2\right)\left[(p+k+l) - m^2\right]} =
$$

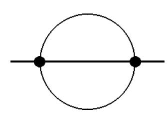
$$
= \frac{i\lambda_R^2 \mu^{8-2n}}{6\,(2\pi)^{2n}} \int d^n k \int d^n l \frac{1}{\left(k^2 - m^2\right)\left(l^2 - m^2\right)\left[(p+k+l) - m^2\right]} =
$$

$$
= \frac{i\lambda_R^2 \mu^{8-2n}}{6\,(2\pi)^{2n}} \pi^n \Gamma\left(3-n\right) \left\{ -\frac{6}{n-4} (m^2)^{n-3} - \frac{p^2}{2} + 3m^2 \right\} \qquad (14.29)
$$

The function $\Gamma(3-n)$ may be written as

$$
\Gamma\left(3-n\right) = \frac{1}{n-4} + \gamma - 1 \qquad (14.30)
$$

As the result, the expression for 3(d) will be

$Fig.3d \quad =$  $=$

$$
= -\frac{i\lambda_R^2 m_R^2}{\left(16\pi^2\right)^2} \left(\frac{m^2}{4\pi\mu^2}\right)^{n-4} \frac{1}{(n-4)^2}
$$

$$
- \frac{i\lambda_R^2}{\left(16\pi^2\right)^2} \frac{1}{(n-4)} \left\{ \frac{p^2}{12} - \frac{m^2}{2} + (\gamma - 1)m^2 \right\} \qquad (14.31)
$$

The complete two loop calculations give us the following expression for $\Gamma^{(2)}$

$$
\Gamma^{(2)} = \frac{p^2 - m^2}{i} - i\delta m^2 - \frac{i\lambda_R m_R^2}{16\pi^2(n-4)} +
$$

$$+\frac{3i}{(16\pi^2)^2}\frac{\lambda_R^2 m_R^2}{(n-4)^2}+\frac{3i}{2(16\pi^2)^2}\frac{\lambda_R^2 m_R^2}{(n-4)}(\gamma-1)$$

$$+\frac{i}{(16\pi^2)^2}\frac{\lambda_R^2 m_R^2}{(n-4)^2}+\frac{i}{2(16\pi^2)^2}\frac{\lambda_R^2 m_R^2}{(n-4)}\gamma$$

$$-\frac{i\lambda_R^2}{(16\pi^2)^2}\frac{m_R^2}{(n-4)^2}-\frac{i\lambda_R^2}{(32\pi^2)^2}\frac{m_R^2}{(n-4)}[4\gamma-2]$$

$$-\frac{i\lambda_R^2 m_R^2}{(16\pi^2)^2}\left(\frac{m-R^2}{4\pi\mu^2}\right)^{n-4}\frac{1}{(n-4)^2}$$

$$-\frac{i\lambda_R^2}{(16\pi^2)^2}\frac{1}{(n-4)}\left\{\frac{p^2}{12}-\frac{m-R^2}{2}+(\gamma-1)m_R^2\right\} \tag{14.32}$$

or

$$i\Gamma^{(2)}=p^2-m^2+\delta m^2+\frac{\lambda_R m_R^2}{16\pi^2(n-4)}-$$

$$-\frac{2}{(16\pi^2)^2}\frac{\lambda_R^2 m_R^2}{(n-4)^2}+\frac{1}{(16\pi^2)^2}\frac{\lambda_R^2}{(n-4)}\left[\frac{p^2}{12}-\frac{m_R^2}{2}\right] \tag{14.33}$$

To make $\Gamma^{(2)}$ finite in the second order of λ_R^2, we will put

$$\delta m^2=m_R^2\left\{\frac{\lambda_R}{(n-4)}b_{11}+\frac{\lambda_R^2}{(n-4)}b_{12}+\frac{\lambda_R^2}{(n-4)^2}b_{22}+O(\lambda_R^3)\right\} \tag{14.34}$$

Then, combining terms in the proper way, we get

$$i\Gamma^{(2)}=\frac{m_R^2}{(n-4)}\left\{b_{11}+\frac{1}{16\pi^2}\right\}+$$

$$\left[1+\frac{1}{12(16\pi^2)^2}\frac{\lambda_R^2}{(n-4)}\right]\times\left\{p^2-m_R^2\left[1+\frac{1}{12(16\pi^2)^2}\frac{\lambda_R^2}{(n-4)}\right]^{-1}\times\right.$$

$$\left.\times\left\{1+\frac{\lambda_R^2}{(n-4)^2}\left[\frac{1}{2(16\pi^2)^2}-b_{12}\right]+\frac{\lambda_R^2}{(n-4)}\left[\frac{2}{(16\pi^2)^2}-b_{22}\right]\right\}\right\} \tag{14.35}$$

It follows from (14.35) that coefficient b_{11} is exactly (14.15).

Two other coefficients are found from the suggestion that $\Gamma^{(2)}$ is analytic at $n = 4$. In the result we get that

$$b_{22}=\frac{1}{2(16\pi^2)^2} \tag{14.36}$$

and b_{12} is the solution of the equation

$$\frac{\lambda_R^2}{(n-4)^2}\left[b_{12}-\frac{5}{12(16\pi^2)^2}\right]=0 \tag{14.37}$$

It gives

$$b_{12}=\frac{5}{12(16\pi^2)^2} \tag{14.38}$$

After these calculations we will have

$$i\Gamma^{(2)}=\left[1+\frac{1}{12(16\pi^2)^2}\frac{\lambda_R^2}{(n-4)}\right]\times\left(p^2-m_R^2\right) \tag{14.39}$$

or

$$\Gamma^{(2)} = Z\Gamma^{(2)}_{reg} \tag{14.40}$$

where (wave function) renormalization constant Z is

$$Z = 1 + \frac{1}{12(16\pi^2)^2}\frac{\lambda_R^2}{(n-4)} \tag{14.41}$$

From the calculations of this section we found that the bare mass m_B and coupling constant λ_B for the second order of perturbation theory are

$$m_B^2 = m_R^2\left\{1 + \frac{\lambda_R}{(n-4)}\left[-\frac{1}{16\pi^2} + \frac{5}{12}\frac{\lambda_R}{(16\pi^2)^2}\right] + \right.$$

$$\left. + \frac{2\lambda_R^2}{(16\pi^2)^2}\frac{1}{(n-4)^2} + O\left(\lambda_R^3\right)\right\} \tag{14.42}$$

and

$$\lambda_B = \mu^{4-n}\left\{\lambda_R - \frac{3}{(16\pi^2)}\frac{\lambda_R^2}{(n-4)} + O\left(\lambda_R^3\right)\right\} \tag{14.43}$$

The expressions (14.42) and (14.43) connect non-renormalizable and renormalizable parameters of the model, and the model is renormalized in two loop approximation of the perturbative regime.

Chapter 15

GREEN'S FUNCTION OF BOSON

IN FINITE TEMPERATURE REGIME

The aim of this chapter is to construct renormalizable Green's function for a boson in a heat bath with a definite temperature. For this purpose we will use the real time representation for the finite temperature propagator that will let us obtain necessary results in a natural and elegant way.

We will repeat calculations for the contributions in self energy of the boson in one and two loop approximations based on the scheme, developed in the previous chapter.

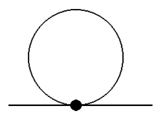
In contrast to chapter XIV we will consider that all internal lines of Feynman graphs are the finite temperature propagators of the form

$$D(k) = D_0(k) + D_\beta(k) = \frac{i}{k^2 - m^2} + \frac{2\pi\delta(k^2 - m^2)}{e^{\beta|k^0|} - 1}, \tag{15.1}$$

where $\beta^{-1} = T$ is the temperature.

Finite temperature calculations don't change Feynmann rules for loop calculations [Morley 1978], [Bernard 1974].

Fig 1b) gives

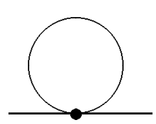
$Fig.1b \quad =$  $=$

$(T \neq 0)$

$$= \frac{1}{2}\frac{(-i\lambda_R\mu^{4-n})}{(2\pi)^n}\int d^n k D(k) =$$

$$= \frac{1}{2}\frac{(-i\lambda_R\mu^{4-n})}{(2\pi)^n}\int d^n k D_0(k) + \frac{1}{2}\frac{(-i\lambda_R\mu^{4-n})}{(2\pi)^n}\int d^n k D_\beta(k)$$

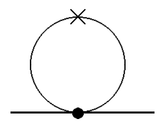
or in a more compact form

$Fig.1b \quad =$  $-$

$(T = 0)$

$$- \frac{1}{2}\frac{(-i\lambda_R\mu^{4-n})}{(2\pi)^n}\int d^n k D_\beta(k). \tag{15.2}$$

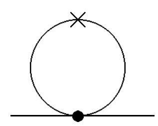
The counterterm Fig. 3a) is

$Fig.3a \quad =$  $=$

$(T \neq 0)$

$$= \frac{1}{2}\frac{(-i\lambda_R\mu^{4-n})}{(2\pi)^n}(-i\delta m^2)\int d^nkD_0^2(k) - \lambda_R\delta m^2\int\frac{d^4k}{(2\pi)^4}D_\beta(k)D_0(k),$$

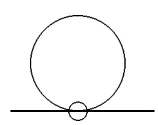
or

$Fig.3a \quad =$  $-$

$(T = 0)$

$$- \lambda_R\delta m^2\int\frac{d^4k}{(2\pi)^4}D_\beta(k)D_0(k). \tag{15.3}$$

The contribution of the counterterm Fig. 3b) is

$Fig.3b \quad =$  $=$

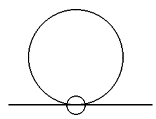
$(T \neq 0)$

$$= \frac{1}{2}\frac{(-i\delta\lambda\mu^{4-n})}{(2\pi)^n}\int d^nkD(k)$$

$$= \frac{1}{2}\frac{(-i\delta\lambda\mu^{4-n})}{(2\pi)^n}\int d^nk(D_0(k) + D_\beta(k)),$$

and

$Fig.3b$ $=$

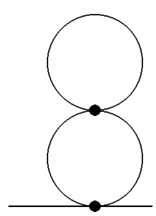


$(T=0)$ $-$

$$- (i\delta\lambda)\int \frac{d^n k}{(2\pi)^n} D_\beta(k). \tag{15.4}$$

Two loop contribution Fig. 3c) may be written in the form

$Fig.3c$ $=$



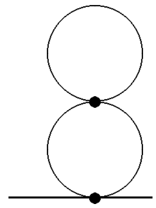
$(T \neq 0)$ $=$

$$= \frac{1}{4}\frac{(-i\lambda_R\mu^{4-n})^2}{(2\pi)^{2n}}\int d^n k\int d^n q (D_0(k)+D_\beta(k))^2(D_0(q)+D_\beta(q))$$

$$= \frac{1}{4}\frac{(-i\lambda_R\mu^{4-n})^2}{(2\pi)^{2n}}\int d^n k\int d^n q\left[D_0^2(k)D_0(q) + D_0^2(k)D_\beta(q)+\right.$$

$$\left.+2D_0(k)D_\beta(k)D_0(q)+2D_0(k)D_\beta(k)D_\beta(q)+D_\beta^2(k)D_0(q)+D_\beta^2(k)D_\beta(q)\right].$$

Then

$Fig.3c \quad =$  $-$

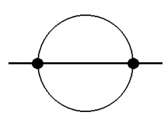
$(T=0)$

$$- \frac{\lambda_R^2}{4} \int \frac{d^4q}{(2\pi)^4} D_\beta(q) \left\{ \frac{\mu^{4-n}}{(2\pi)^n} \int d^n k D_0^2(k) \right\}$$

$$- \frac{\lambda_R^2}{2} \left\{ \int \frac{d^4q}{(2\pi)^4} D_0(k) D_\beta(k) \right\} \left\{ \frac{\mu^{4-n}}{(2\pi)^n} \int d^n q D_0(q) \right\}$$

$$- \frac{\lambda_R^2}{2} \left\{ \int \frac{d^4q}{(2\pi)^4} D_\beta(q) \right\} \left\{ \int \frac{d^4k}{(2\pi)^4} . D_0(k) D_\beta(k) \right\} \tag{15.5}$$

Finally the contribution Fig. 3d) will be

$Fig.3d \quad =$  $=$

$(T \neq 0)$

$$\frac{1}{6} \frac{(-i\lambda_R \mu^{4-n})^2}{(2\pi)^{2n}} \int d^n k D_\beta(k) \times$$

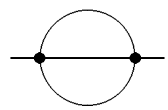
$$\times \left[\int d^n q D_\beta(q) D_0(q-p-k) + \int d^n q D_\beta(q) D_0(k-p-q) \right.$$

$$+ \int d^n q D_\beta(q) D_0(q+p+k) + \int d^n q D_\beta(q) D_0(q-k+p)$$

$$+\int d^n q D_0(q) D_0(q-p-k) + \int d^n q D_0(q) D_0(q+p+k)\Big]$$

or

$$Fig.3d \quad =$$



$$+$$

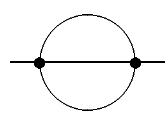
$(T=0)$

$$+\frac{1}{2}\frac{(-i\lambda_R \mu^{4-n})^2}{(2\pi)^{2n}}\int d^n k D_\beta(k)\times$$

$$\times\left[\int d^n q D_\beta(q) D_0(q+p+k) + \int d^n q D_0(q) D_0(k+p+q)\right]$$

So, we get

$$Fig.3d \quad =$$



$$-$$

$(T=0)$

$$-\frac{\lambda_R^2}{2}\int \frac{d^4 k}{(2\pi)^4} D_\beta(k)\left\{\int \frac{d^4 q}{(2\pi)^4}.D_\beta(q) D_0(k+q+p)\right\}$$

$$-\frac{\lambda_R^2}{2}\int \frac{d^4 k}{(2\pi)^4} D_\beta(k)\left\{\frac{\mu^{4-n}}{(2\pi)^n}\int d^n q D_0(q) D_0(k+q+p)\right\} \qquad (15.6)$$

Now we can find counterterms.

Assuming the sum of the finite temperature contributions of the equations (15.3) and (15.5) zero,

$$\left\{\lambda_R \delta m^2 + \frac{\lambda_R^2}{2}\frac{\mu^{4-n}}{(2\pi)^n}\int d^n q D_0(q)\right\}\int \frac{d^4k}{(2\pi)^4} D_0(k) D_\beta(k) = 0.$$

we find the expression for δm^2 in the form

$$\delta m^2 = -\frac{\lambda_R}{2}\frac{\mu^{4-n}}{(2\pi)^n}\int d^n q D_0(q). \tag{15.7}$$

The divergent part of this counterterm will be

$$\delta m^2_{div} = -\frac{\lambda_R}{16\pi^2}\frac{m_R^2}{(n-4)}. \tag{15.8}$$

The following counterterm $\delta\lambda$ may be found by the summation of the finite temperature contributions of the equations (15), (15.5) and (15.6)

$$\int \frac{d^4k}{(2\pi)^4} D_\beta(k)\left\{i\delta\lambda + \frac{\lambda_R^2}{2}\left[\frac{\mu^{n-4}}{(2\pi)^n}\int d^n q D_0^2(q)\right]\right.$$

$$\left. +\lambda_R^2\left[\frac{\mu^{n-4}}{(2\pi)^n}\int d^n q D_0(q) D_0(k+q+p)\right]\right\} = 0.$$

For zero external momentum p the divergent part of the $\delta\lambda$ will be determined by the divergent part of the integral

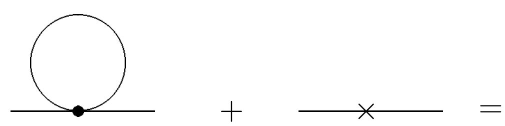
$$\delta\lambda = \frac{3}{2}(i\lambda_R^2)\left[\frac{\mu^{n-4}}{(2\pi)^n}\int d^n q D_0^2(q)\right] \tag{15.9}$$

and the divergent contribution will be

$$\delta\lambda_{div} = -\frac{3\lambda_R^2}{16\pi^2}\frac{1}{(n-4)}. \tag{15.10}$$

The temperature counterterms (15.8) and (15.10) have the same structure as the counterterms which annihilate the divergent parts of zero temperature loop contributions.

It is easy to see that at $T = 0$ the sum of the loop contribution and the counterterm gives zero:



$$= \frac{(-i\lambda_R\mu^{4-n})}{2(2\pi)^n}\int d^nkD_0(k) - \delta m^2 = 0.$$

It leads us to the equation (15.7).

Loop contributions and counterterm at $T = 0$ in $\Gamma^{(4)}$ (Fig.II-2) gives the equation

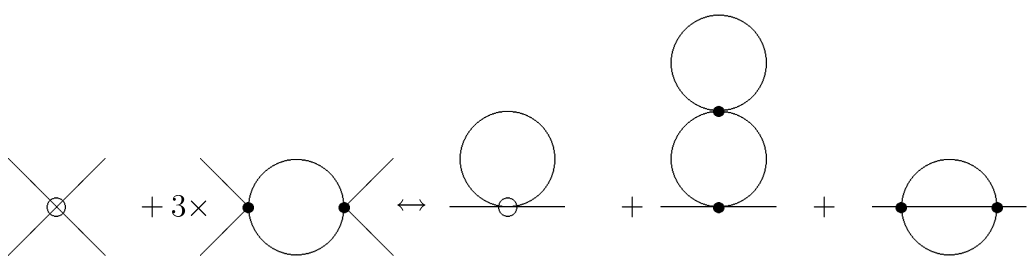
$$\frac{3}{2}\frac{\lambda_R^2\mu^{4-n}}{(2\pi)^n}\int d^nkD_0(k)D_0(k+p) - i\delta\lambda = 0,$$

which coincides with (15.9).

From the mentioned above analysis we can conclude that the following loops have the same divergent structure:

$T = 0$ $T \neq 0$

and



$T = 0$ $T \neq 0$

The Green's function $D'(p)$ of the boson is the object which takes into account virtual processes of creation and anihilation of the additional particles when this boson moves through the vacuum.

The graph composing $D'(p)$ may be divided into two distinct and unique classes of proper and improper graphs (Fig.II-4)[1].

[1]The proper graphs cannot be divided into two disjoint parts by the removal of a single line, whereas the improper ones can be disjoint [Bjorken & Drell 1965]

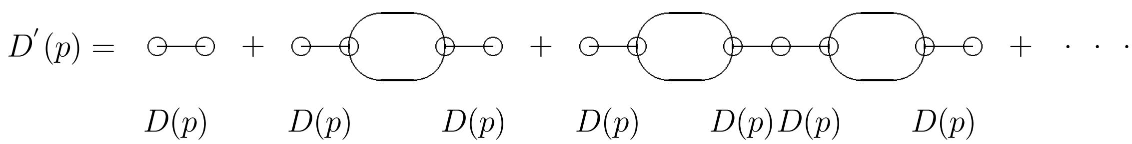


Fig.II-4 Green's function $D^{'}(p)$ of the boson as sum of proper self-energy insertions.

In accordance with Fig.II-4 the Green's function $D^{'}(p)$ is obtained by the summing of the series:

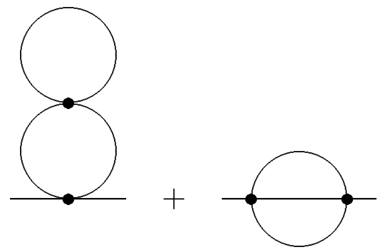
$$D^{'}(p) = D(p) + D(p)\left(\frac{\Sigma(p)}{i}\right)D(p)$$

$$+ D(p)\left(\frac{\Sigma(p)}{i}\right)D(p)D(p)\left(\frac{\Sigma(p)}{i}\right)D(p) + ...$$

$$= D(p)\frac{1}{1 + i\Sigma(p)D(p)} = \frac{i}{p^2 - m^2 - \Sigma(p)} \qquad (15.11)$$

In this equation $\Sigma(p)$ is the sum of all two point improper graphs. All divergent improper graphs and their counterterm graphs (Fig.II-1,II-3) may be divided into two parts of the first and the second orders with respect to λ_R. We will define them as Σ_1 and Σ_2 self-energy graphs.

One can write finite contributions in Σ_2 in the form

$(-i)\Sigma_2 =$  $=$

$$= -\frac{\lambda_R^2}{2}\int \frac{d^4q}{(2\pi)^4} D_\beta(q) \left\{ \int d^4k D_\beta(k) D_0(k+q+p) \right\}$$

$$-\frac{\lambda_R^2}{2}\int \frac{d^4q}{(2\pi)^4} D_\beta(q) \left\{ \frac{\mu^{4-n}}{(2\pi)^n} \int d^4k D_0(k) D_0(k+q+p) \right.$$

$$\left. -\frac{\mu^{4-n}}{(2\pi)^n} \int d^4k D_0(k) D_0(k+p) \right\}_{p=0} \tag{15.12}$$

Let us introduce functions

$$F_\beta = \frac{1}{2}\int \frac{d^4q}{(2\pi)^4} D_\beta(q) \tag{15.13}$$

and

$$iI_\beta(p) = \int \frac{d^4q}{(2\pi)^4} D_\beta(q) D_0(q+p). \tag{15.14}$$

Then we get

$$\Sigma_2 = \lambda_R^2 F_\beta I_0(0) + \frac{\lambda_R^2}{2} G_\beta \tag{15.15}$$

where

$$G_\beta = (F_\beta, (I_\beta + I_0))_{finite} \tag{15.16}$$

The first equation in (15.16) is a scalar product of the form

$$(F_\beta, I_\beta) = \int \frac{d^4k}{(2\pi)^4} D_\beta(k) \int \frac{d^4q}{(2\pi)^4} D_0(q) D_\beta(k+q) \tag{15.17}$$

and the following one is

$$(F_\beta, I_0)_{finite} = \int \frac{d^4k}{(2\pi)^4} D_\beta(k) \left\{ \int \frac{d^4q}{(2\pi)^4} D_0(q) D_0(k+q) \right\}_{finite} \tag{15.18}$$

The contribution of the first order in Σ_1 may be found from (15.2).

This contribution is

$$\Sigma_1 = \frac{\lambda_R}{2} \int \frac{d^4q}{(2\pi)^4} D_\beta(q) = \lambda_R F_\beta. \tag{15.19}$$

The temperature contribution in the boson's mass[2] will have the following form

$$m^2(T) = m_R^2 + \Sigma_1 + \Sigma_2$$

[2]For finite temperature quantum electrodynamics the non-equality of fermionic masses $\delta m_\beta / m \sim \alpha (T/m)^2$ is described by a similar equation [Peressutti & Skagerstam 1982]

$$= m_R^2 + \lambda_R F_\beta + \lambda_R^2 F_\beta I_\beta(0) + \frac{\lambda_R^2}{2}\lambda_R^2 G_\beta \tag{15.20}$$

As the result the Green's function will be

$$D^{'}(p) = \frac{i}{p^2 - m^2(T)} \tag{15.21}$$

Thus we have computed the finite temperature Green's function of a boson in two-loop approximation in the form of Feynman propagator with finite temperature dependent mass parameter (15.20).

Chapter 16

THERMAL PROPERTIES OF BOSON

In chapter XV we showed that the model is renormalizable in each order of the perturbative regime, and found the finite temperature propagator of a boson in a heat bath in two loop approximation. We also got the expression for the finite temperature mass of a boson. These results may help us to get an effective Hamiltonian of the particle and to study its finite temperature behavior in gravitational fields.

16.1 Effective Hamiltonian of the boson in non-relativistic approximation

After renormalization the pole of boson propagator (15.21) may be written as

$$E = \left[\vec{p}^{\,2} + m_R^2 + \lambda_R F_\beta + \lambda_R^2 F_\beta I_\beta(0) + \frac{\lambda_R^2}{2} \lambda_R^2 G_\beta \right]^{\frac{1}{2}} \tag{16.1}$$

We can rewrite the equation (16.1) in non-relativistic approximation in the following form

$$E = m_R \left\{ 1 + \lambda_R f(\beta m_R) + o(\lambda_R^2) \right\}^{\frac{1}{2}} \left[1 + \frac{\vec{p}^{\,2}}{m_R^2 \left\{ 1 + \lambda_R f(\beta m_R) + o(\lambda_R^2) \right\}} \right]^{\frac{1}{2}}$$

$$= m_R + \frac{1}{2} \lambda_R m_R f(\beta m_R) + \frac{\vec{p}^{\,2}}{2 m_R} \left(1 + \frac{1}{2} \lambda_R f(\beta m_R) \right)^{-1} + o(\lambda_R^2) \tag{16.2}$$

Here the function $f(y)$ is connected with the function $F_\beta(y)$ (15.13) in the following way

$$F_\beta(\beta m_R) = \frac{1}{2} \int \frac{d^3 k}{(2\pi)^3} \frac{1}{\varepsilon \left(e^{\beta \varepsilon} - 1 \right)} = m_R^2 f(\beta m_R)$$

The function $f(y)$ has an asymptotic form (for $(y \ll 1)$) [Dolan & Jackiw 1974]:

$$f(y) = \frac{1}{(2\pi)^2} \int_1^\infty dx \frac{\sqrt{x^2 - 1}}{e^{xy} - 1}$$

$$= \frac{1}{24 y^2} - \frac{1}{8 \pi y} + O\left(y^2 \ln y^2 \right), \tag{16.3}$$

which is very useful for the analysis of the high temperature behavior of the model.

16.2 Inertial and gravitational masses of a boson

For our following calculations we will consider that the quantum system interacts with the gravitational field which is described by the metric

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{\Phi}{2}\delta_{0\mu}\delta_{0\nu}, \tag{16.4}$$

where Φ is a gravitational potential.

The second term in (16.4) describes a small correction to the Minkowski metric which is connected with the presence of the gravitational field
[Landau & Lifshitz], [Misner et al. 1973].

In order to write the Hamiltonian of the boson in the presence of gravitational field one can consider that temperature T changes according to Tolmen's law [Tolmen 1934].

$$T = \frac{T_0}{1+\Phi}, \tag{16.5}$$

where T_0 is the temperature with $\Phi = 0$.

The finite temperature Hamiltonian with precision to the first leading term of the series (16.3) will be

$$H = \frac{\vec{p}^{\,2}}{2}\left(m_R + \frac{\lambda_R}{48}\frac{T_0^2}{(1+\Phi)^2 m_R}\right)^{-1} + m_R + \frac{\lambda_R}{48}\frac{T_0^2}{m_R(1+\Phi)^2} + m_R\Phi + ... \tag{16.6}$$

The last term of the equation (16.6) describes energy of boson's interaction with gravitational field.

Let us rewrite the Hamiltonian (16.6) as

$$H = \frac{\vec{p}^{\,2}}{2}\left(m_R + \frac{\lambda_R}{48}\frac{T_0^2}{(1+\Phi)^2 m_R}\right)^{-1} + \left(m_R - \frac{\lambda_R}{24}\frac{T_0^2}{m_R}\right)\Phi + ... \tag{16.7}$$

The acceleration of the boson in a gravitational field may be found from the quantum mechanical relation

$$\vec{a} = -\,[H,[H,\vec{r}]] = -\left(1 - \frac{\lambda_R}{12}\frac{T_0^2}{m_R^2}\right)\nabla\Phi, \tag{16.8}$$

and the mass ratio will be

$$\frac{m_g}{m_i} = 1 - \frac{\lambda_R}{12}\frac{T_0^2}{m_R^2} \tag{16.9}$$

so the inertial and gravitational masses of the boson in the heat bath are seen to be unequal.

One can estimate the value of $\delta m_g/m_i$ for some gravitational source. Let the source of gravitational field be the Sun ($1.989 \times 10^{30} kg$) then the relation (16.9) for the combined boson (Cooper pair with mass $m_b = 1Mev$) in the heat bath with temperature $300K$ gives the following corrections for non-equality between masses

$$\frac{\delta m_g}{m_i} = \frac{\lambda_R}{12}\frac{T_0^2}{m_R^2} \sim \lambda \times 10^{-17} \tag{16.10}$$

or

$$10^{-21} < \frac{\delta m_g}{m_i} < 10^{-17} \tag{16.11}$$

for the range of the coupling constant $10^{-3} < \lambda < 10^{-2}$.

From the analysis we made in this chapter one may conclude that thermal interaction of the bosons in a gravitational field causes non-equality between inertial and gravitational masses. Non-thermal systems do not demonstrate such properties.

The calculations for non-equality between inertial and gravitational mass of electron were made for thermal quantum electrodynamics by Donoghue [Donoghue et al. 1984]. His result for massive fermions has the same functional structure as the equation (16.9).

PART III

NON-LINEAR MODELS IN TOPOLOGY NON-TRIVIAL SPACE-TIME

Introduction

The usual method of generating spontaneous symmetry breaking in quantum field theory is to introduce multiplets of vector or scalar fields which develop a nonvanishing vacuum expectation values [Goldstone 1961]. This mechanism is not necessary. The general features of spontaneous symmetry breaking are independent of whether the Goldstone or Higgs particle is associated with an elementary field or with a composite field. However it is possible to develop a dynamical theory of elementary particles in which the origin of the spontaneous symmetry breaking is dynamical [Nambu & Jona-Lasinio 1961]. The finite mass appears in analogy with the phenomenon of superconductivity. This part III deals with the research of [Bender et al.1977], [Tamvakis & Guralnik], [Kawati & Miyata 1981] in the field of dynamical symmetry breaking and phase transitions. Models with dynamical mass generation are interesting for the following reasons:

First: Initially massless fermi fields expose γ_5 invariance, which is violated by the dynamical mass generation. Thus the initial symmetry will be broken, and this broken symmetry may appear in experimental measurements [Taylor 1976].

Second: Dynamical mass of particles in such models is a function of a constant of interaction of primary fields. Perhaps this will indicate how to solve the puzzle of the origin of particle mass.

Third: Higgs mechanism introduces into the theory additional parameters which are connected with Higgs-Goldstone fields. These are additional difficulties of the model. [Kaku 1988].

Distinctive and important characteristics of the models with dynamic symmetry violation are the introduction of bound states of particles, and calculations in non-perturbative regime. Estimations of the condensate may be obtained from the sum rule [Floratos et al. 1984], or from Monte-Carlo method [Binder 1979], or for the semiclassical instanton solutions of the Yang-Mills equations [Shyriak 1983].

To develop further the theory one should take into account also the topology of space-time [Toms 1980], [Isham 1978], [Ford 1980]. One should consider 3-D and 4-D spinor models with non-linearity of the type $(\bar{\psi}\psi)^2$. The non-linear Gross-Neveu model in 3-D space-time in non-trivial topology is studied in chapter XVII. 3-D Heisenberg-Ivanenko model in non-Euclidean space-time and 4-D Heisenberg-Ivanenko model in Riemann space-time are studied in chapter XVIII. The influence of topology and curvature on the generation of dynamical mass of fermions and the effects of violation of symmetry in these models are also studied in chapter XVIII.

Chapter 17

NON-PERTURBATIVE EFFECTS

IN GROSS-NEVEU MODEL

Quantum Field Theory may be essentially symplified in the limit of high interanal symmetries such as O(N), SU(N) and so on. In some appropriate cases field models can be solved strictly in the limit of large flavor numbers N [Gross & Neveu 1974]. Asymptotically free Gross-Neveu model without dimensional parameters in the Lagrangian is one of such models. This model is renormalizable in 3-D dimensions. The solution of this model shows that the phenomenon of dimensional transmutation [Coleman & Weinberg 1974] has place and there is a gap in mass spectrum of the model. Studying this model one will find the connection between topological characteristics of space time and the behavior of the solution of the mass gap equation and also the behavior of dinamical mass of the model. N-flavor Gross-Neveu model is described by the Lagrangian

$$L = \bar{\psi}_i i\hat{\partial}\psi_i + \frac{g^2}{2}\left(\bar{\psi}_i\psi_i\right)^2 \tag{17.1}$$

This Lagrangian is invariant under the discrete transformations:

$$\psi \to \gamma_5\psi, \quad \bar{\psi} \to -\bar{\psi}\gamma_5 \tag{17.2}$$

The generating functional of the model

$$Z[\eta,\bar{\eta}] = \int D\psi D\bar{\psi} \exp\left[i\int d^n x \left(i\bar{\psi}\partial\psi + (1/2)g^2(\bar{\psi}\psi)^2 + \bar{\eta}\psi + \bar{\psi}\eta\right)\right]$$

may be rewritten in the form:

$$Z[\eta,\bar{\eta},\sigma] = \int D\psi D\bar{\psi} D\sigma$$

$$\times \exp\left[i\int d^n x \left(i\bar{\psi}\partial\psi - g(\bar{\psi}\psi)\sigma - (1/2)\sigma^2 + \bar{\eta}\psi + \bar{\psi}\eta\right)\right] \tag{17.3}$$

Here we introduced the new field σ and used a useful relation

$$\int D\sigma \exp\left[-i(1/2)\left(\sigma,\sigma\right) - i\left(g\bar{\psi}\psi,\sigma\right)\right] \propto \exp\left[(i/2)g^2\left(\bar{\psi}\psi\right)^2\right] \tag{17.4}$$

Since

$$m\bar{\psi}\psi \to m\left(-\bar{\psi}{\gamma_5}^2\psi\right) = -m\bar{\psi}\psi \tag{17.5}$$

we must put $m = 0$ for symmetry of the model.

Therefore a new Lagrangian may be written in the form

$$L_\sigma = \bar{\psi}_i i\hat{\partial}\psi_i - \sigma\left(\bar{\psi}_i\psi_i\right) - \frac{1}{2g^2}\sigma^2 \tag{17.6}$$

and the symmetry of (17.6) is

$$\psi \to \gamma_5\psi, \quad \bar{\psi} \to -\gamma_5\bar{\psi}, \quad \sigma \to -\sigma \tag{17.7}$$

The generating functional of the model after integration over the matter fields will be

$$Z[0] = \int D\bar{\psi}D\psi D\sigma \exp i\int L_\sigma(x)dx$$

$$= \int D\sigma\cdot Det(i\hat{\partial} - \sigma)\exp[\frac{-i}{2g^2}(\sigma,\sigma)] \tag{17.8}$$

Then effective action is written as

$$\Gamma[\sigma] = -i\ln Det(i\hat{\partial} - \sigma) - \frac{1}{2g^2}(\sigma,\sigma) \tag{17.9}$$

and the effective potential is

$$V_{eff} = \frac{iN}{2}(\mathrm{Tr}\hat{1}) \int \frac{d^2k}{(2\pi)^2} \ln\left(k^2 - \sigma^2\right) + \frac{1}{2g^2}\sigma^2 \tag{17.10}$$

The conditions for the energy to be minimal are

$$\left(\frac{\delta V_{eff}}{\delta\sigma}\right)_{|\sigma=\sigma_c} = 0, \quad \left(\frac{\delta^2 V_{eff}}{\delta\sigma^2}\right)_{|\sigma=\sigma_c} > 0 \tag{17.11}$$

From these conditions one can get the gap equation for definition of σ_c (σ_c defines the minimum of the effective potential) in the form

$$\frac{1}{\lambda} = \mathrm{Tr}\hat{1} \int \frac{d^2\bar{k}}{(2\pi)^2} \frac{1}{\bar{k}^2 + \sigma_c^2} \tag{17.12}$$

where constant $\lambda = g^2 N$.

17.1 Trivial case. Euclidean space time.

Let us find the solution of the gap equation (17.12).

Ultraviolet cut-off of the integral gives:

$$\int_{-\Lambda}^{\Lambda} \frac{d^2\bar{k}}{(2\pi)^2} \frac{1}{\bar{k}^2 + \sigma_c^2}$$

$$= (1/2\pi) \int_0^{\Lambda} \frac{2\pi dk^2}{\bar{k}^2 + \sigma_c^2} = (1/4\pi) \ln \frac{\Lambda^2}{\sigma^2} \tag{17.13}$$

Then, after regularization of (17.13) we get the equation

$$\frac{1}{\lambda(\Lambda)} = \frac{1}{2\pi} \ln \frac{\Lambda^2}{\sigma^2} \tag{17.14}$$

Let the subtraction point be $\mu = \sigma$, then the renormalized coupling constant may be written as

$$\frac{1}{\lambda(\mu)} = \frac{1}{2\pi} \ln \frac{\mu^2}{\sigma^2} \tag{17.15}$$

To eliminate the parameter μ one can use the methods of the renormalization group. The β-function in one loop approximation is

$$\beta(\lambda_R(\mu)) = -\frac{1}{\pi} (\lambda_R(\mu))^2 \tag{17.16}$$

then Gell-Mann Low equation

$$\frac{d\lambda_R}{\lambda_R^2} = -\frac{1}{\pi} \frac{d\xi}{\xi} \tag{17.17}$$

with initial condition $\lambda(\xi_0) = \lambda_0$ determines the behavior of the coupling constant with respect to scaling of the momentum:

$$\lambda_R(t) = \frac{\lambda_0}{1 + (\lambda_0/\pi)t} \tag{17.18}$$

where $t = \ln(\xi/\xi_0)$

The dynamical mass can be found from (17.11) in the form

$$\sigma_c(triv.) = \mu \exp(-\frac{\pi}{\lambda_R(\mu)}) = \mu \exp\left(-\frac{\pi}{\lambda_0}\right) = const. \tag{17.19}$$

17.2 Non-trivial topology of space time.

In this case the constant of interaction can be written as

$$\frac{1}{\lambda(\mu, L)} = \frac{1}{2\pi}\left[\ln\frac{\mu^2}{\sigma_c^2} + f(L, \sigma_c)\right] \tag{17.20}$$

where $f(\sigma_c, L)$ is some function which depends on topological parameter L.

The β function is

$$\beta(\lambda_R(\mu, L)) = -\frac{1}{\pi}(\lambda_R(\mu, L))^2 \tag{17.21}$$

and the solution of Gell-Mann-Low equation with $\lambda_R(\xi_0, L) = \lambda_0(L)$ is expressed by the equation:

$$\lambda_R(t, L) = \frac{\lambda_0(L)}{1 + (\lambda_0(L)/\pi)t} \tag{17.22}$$

The dynamical mass is governed by the equation

$$\sigma_c = \mu \exp\left(-\frac{1}{\lambda_0(\mu, L)} - \frac{f(\sigma_c, L)}{2}\right) \tag{17.23}$$

or

$$\sigma_c = \sigma_c(triv) \exp\left(-\frac{f(\sigma_c, L)}{2}\right) \tag{17.24}$$

One can see that the dynamical mass depends on function $f(\sigma_c, L)$.

The explicit expression of the function $f(\sigma_c, L)$ is

$$f_{(\pm)}(L, \sigma_c) = \pm \frac{1}{\pi^2} \int_0^\infty dx \frac{1}{\sqrt{x^2 + (L\sigma_c)^2}} \left(\exp\sqrt{x^2 + (L\sigma_c)^2} - 1\right)^{-1} \tag{17.25}$$

for topologies of the cylinder (+) and the Mobius strip (−). Therefore non-Euclidean structure of space time leads to redefinition of the gap equation (17.12) for the Gross-Neveu model. That gives us the possibility to define the dependence of the fermionic mass on topology of the space time. Dynamical violation of the γ_5 symmetry occurs when σ is not equal zero. The method developed above is very useful when the number of "flavors" of the fundamental fields is big ($N \to \infty$). This method does not work for a small "flavor" number N. In Quantum Field Theory there is another method, based on an analogy with superconductivity. This is a Mean Field Method [Kawati & Miyata 1981], [Zinn-Justin 1989], which works very well for any number of "flavors" of the particles. The idea of the method is based on effective potential[1] cal-

[1]Effective potential is the generating functional for (1PI) Green's functions [Jackiw 1974, Rivers 1990]

culations, that allows us to take into consideration the effects of topology [Toms 1980] and curvature for self-interacting and gauge models [Ishikawa 1983].

In this work the Mean Field Method is used in dynamical modeling of the behavior of elementary particles and is based on the idea that the masses of compound particles (e.g. nucleons) are generated by the self-interaction of some fundamental fermion fields through the same mechanism as superconductivity. Here the combined particles are treated as the quasi-particles excitations. The Mean Field Method also leads to the mass gap equation, and the solution gives the dynamical mass of the particle. In the following section we will treat the problem of non-Euclidean space-time structure in models with a dynamical mass.

Chapter 18

$(\bar{\psi}\psi)^2$ NON-LINEAR SPINOR MODELS

18.1 Dynamical mass and symmetry breaking

In the construction of the unified models of the elementary particles one can admit the possibility that the mass of combined particles appears as the result of self-interaction of certain fundamental fields, for instance, quarks and leptons from preons, or compound fermions in technicolor models [Rizzo 1983, Terazawa 1980], [Ito & Yasue 1984], [Farhi & Susskind 1980]. Following this idea we can get the gap equation. Its solution can predict the dynamical mass of the compound particles. As in the previous case of Gross-Neveu model, we will study here the effects of non-trivial topology and, also, geometry of background space-time.

In this section we will consider the phenomenon of the dynamical generation of mass of particles in the application to the models in non-trivial space-time.

Let us consider Heisenberg-Ivanenko [Ivanenko 1959] non-linear spinor model with Lagrangian

$$L = \bar{\psi} i \hat{\partial} \psi + \frac{g_0^2}{2\mu_0^2} \left(\bar{\psi} \psi \right)^2 \tag{18.1}$$

where g_0^2 is a massless parameter and parameter μ_0^2 has dimension which is connected with the dimension of space time. The symmetry of the Lagrangian (18.1) is

$$\psi \to \gamma_5 \psi, \quad \bar{\psi} \to -\bar{\psi} \gamma_5 \tag{18.2}$$

We can rewrite this Lagrangian in the new form

$$L_\sigma = \bar{\psi} i \hat{\partial} \psi - g_0 \sigma \left(\bar{\psi} \psi \right) - \frac{\mu_0^2}{2} \sigma^2 \tag{18.3}$$

The symmetry of the last one is (17.7).

The equation of motion for the σ field is

$$\sigma = -\frac{g_0}{\mu_0^2} \left(\bar{\psi} \psi \right) \tag{18.4}$$

thus we may assume that the field σ is a collective field.

Let us consider that σ is $\sigma \to \sigma + \tilde{\sigma}$, where

$$\sigma = (g_0/{\mu_0}^2) < \psi\bar{\psi} > \neq 0$$

is the background field and $\tilde{\sigma}$ is quantum fluctuations of the σ-field.

Then

$$L_\sigma = \bar{\psi}(i\hat{\partial} - g_0\sigma)\psi - g_0\left(\bar{\psi}\tilde{\sigma}\psi\right) - \frac{\mu_0^2}{2}\sigma^2 - \frac{\mu_0^2}{2}\tilde{\sigma}^2 - {\mu_0}^2\sigma\tilde{\sigma} \tag{18.5}$$

The last term of (18.5) may be eliminated because of the redefinition of the sources of the quantum field $\tilde{\sigma}$, and the Feynman graphs will be

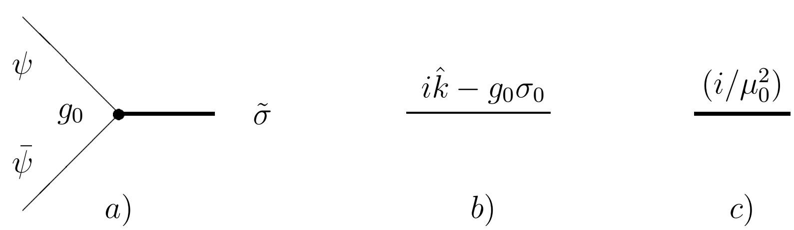


Fig. III-1 Feynman graphs including a collective field.

Graph 1a) describes the interaction of fermi field with collective field, 1b) is the propagator of fermi field, and 1c) is the propagator of collective field.

In tree approximation with respect to σ-field we can write an effective action:

$$\Gamma_{eff}[\sigma] = (-i/2)\ln Det(i\hat{k} - g_0\sigma) - \frac{\mu_0^2}{2}(\sigma, \sigma) \tag{18.6}$$

The effective potential of this model will be

$$V_{eff} = \frac{i}{2}(\mathrm{Tr}\hat{1}) \int \frac{d^n k}{(2\pi)^n} \ln\left(k^2 - (g_0\sigma)^2\right) + \frac{\mu_0^2}{2}\sigma^2 \tag{18.7}$$

Minimum V_{eff} gives the gap equation

$$\sigma_c = (g_0/\mu_0)^2 \sigma_c I\left(g_0\sigma_c\right) \tag{18.8}$$

or, in another form,

$$m = s\lambda_0 m I\left(m\right) \tag{18.9}$$

where m is the dynamical mass of fermionic field $m = g_0\sigma_c$, s is the dimension of γ-matrices and $\lambda_0 = (g_0^2/\mu_0)^2$

Now one can solve the equation (18.9) for different space time topologies.

18.2 Model with topologies $R_1 \times R_1 \times S_1$ and $R_1 \times Mobius\ strip$

The solution of this gap equation is connected with the calculation of the function I(m) of the equation (18.9).

Let us consider two types of topologies:

1) $R_1 \times R_1 \times S_1$ with $\psi(x, y, 0) = \psi(x, y, L)$

and

2) $R_1 \times Mobius\ strip$ with $\psi(x,y,0) = -\psi(x,y,L)$ (Fig.III- 2)

We can find for 3-D space time that

$$I(m) = \Lambda - F_{(\pm)}(L,m) \tag{18.10}$$

where

$$F_{(\pm)}(L,m) = \frac{1}{\pi L}\ln(1 \mp e^{-Lm}) \tag{18.11}$$

with (+) for $R_1 \times R_1 \times S_1$ topology and (-) for $R_1 \times Mobius\ strip$ topology.

The gap equation will be

$$m = m\lambda_0\Lambda\left(1 - \frac{1}{\pi L\Lambda}\ln(1 \mp e^{-Lm})\right) \tag{18.12}$$

The analysis of this expression can be made by the theory of bifurcations [Hale & Kocak 1991]. For this purpose let us write the gap equation (18.12) in the form

$$m = f_{(\pm)}(m,\bar{\lambda}) \tag{18.13}$$

where the functions $f_{(\pm)}(m,\bar{\lambda})$ for topologies $(+)$and $(-)$ are

$$f_{(\pm)}(m,\bar{\lambda}) = m\bar{\lambda}\left(1 - \frac{1}{\pi L\Lambda}\ln(1 \mp e^{-Lm})\right) \tag{18.14}$$

The equation (18.13) has stable m=0 solutions, if $f_m(0, \bar{\lambda}) < 1$.

For $f_m(0, \bar{\lambda}) > 1$ the equation (18.13) has no stable trivial solutions.

One can see that there is a stable m=0 solution for (-) topology with the critical parameter

$$L_c = \frac{\ln 2}{\pi} \frac{\lambda_0}{\lambda_0 \Lambda - 1} \approx \frac{\ln 2}{\pi \Lambda} \tag{18.15}$$

for $\lambda_0 \Lambda = \bar{\lambda} > 1$

The gap equation for topology (+) has no stable trivial solutions.

The dynamical mass in this case is a smooth function with respect to parameter L:

$$m_{(+)} = -\frac{1}{L} \ln \left(1 - f(\Lambda) \exp \left(-\frac{\pi L}{\lambda_0} \right) \right) \tag{18.16}$$

for $\lambda_0 \Lambda = \bar{\lambda} < 1$

The solution for topology $R_1 \times Mobius\ strip$ is

$$m_{(-)} = -\frac{1}{L} \ln \left(\exp \left(\frac{L}{L_c} \ln 2 \right) - 1 \right) \tag{18.17}$$

The restoration of the symmetry takes place when $L = L_c$. In this case the condensate function of the bound state equals zero

$$\sigma_c = \frac{g_0}{\mu_0^2} < \psi \bar{\psi} >= 0 \tag{18.18}$$

Now we can consider the models with the more complicated space-time structures of Klein bottle and Torus topologies.

18.3 Torus topology $M_{3)} = R_1 \times S_1 \times S_1$ and topology $M_{3)} = R_1 \times Klein\ bottle$

We can introduce topologies $R_1 \times S_1 \times S_1$ and $R_1 \times Mobius\ strip$ as an identification of space points for wave functions Fig. III-3:

$$\psi(x,0,0) = \psi(x,L,L)\ and\ \psi(x,0,0) = -\psi(x,L,L) \tag{18.19}$$

For simplicity consider that these topologies have only one parameter $L = L^{'}$. The gap equations for the topologies will be

$$m = m\bar{\lambda}s\left\{1 + \sqrt{m/(\Lambda^2 L)}(\pi)^{-3/2}[\sum_{n=1}^{\infty}\frac{1}{\sqrt{2n}}K_{-1/2}(2Lmn)\right.$$
$$\left.+\sum_{n,k=1}^{\infty}K_{-1/2}(2Lm\sqrt{n^2+k^2})\left(n^2+k^2\right)^{-1/4}]\right\} \tag{18.20}$$

and

$$m = m\bar{\lambda}s\left\{1 + \sqrt{m/(\Lambda^2 L)}(\pi)^{-3/2}[\sum_{n=1}^{\infty}\frac{1}{\sqrt{2n}}K_{-1/2}(2Lmn)\right.$$
$$\left.+\sum_{n,k=1}^{\infty}(-1)^k K_{-1/2}(2Lm\sqrt{n^2+k^2})\left(n^2+k^2\right)^{-1/4}]\right\} \tag{18.21}$$

The solutions of (18.20) and (18.21) give dynamical masses for these topologies [Kulikov & Pronin 1989].

The results of this paragraph show that the application of the Mean Field Method to non-linear models gives non-renormalized solutions, though we can obtain some information about the influence of topology on dynamical mass behavior. The evaporation of condensate and restoration of chiral symmentry proceed in different ways and are ruled by topology. There are topologies in which these phenomena do not take place. We believe that these results are important in the bag models, because the energy of bag is dependent on non-perturbative effects and boundary conditions [Guidry 1991].

18.4 Non-linear spinor $(\bar{\psi}\psi)^2$ model in Riemann space-time at finite temperature.

In this section we will treat the problem of dynamical mass generation of the non-linear spinor model in 4-D Riemann space-time at finite temperature. We will get the finite temperature effective potential and find out information about the influence of curvature of the background gravitational field and temperature on the value of dynamical fermionic mass.

Let the total Lagrangian of the model be

$$L = L_g + L_m \tag{18.22}$$

where the gravitational Lagrangian is[1]

[1]For clear understanding of the problem, we study the fermionic system in a weak gravitational field.

$$L_g = \frac{1}{16\pi G_0}(R - 2\tilde{\Lambda}_0) \tag{18.23}$$

and the Lagrangian of matter field L_m is

$$L_m = \bar{\psi} i \bar{\gamma}^\mu \nabla_\mu \psi + \frac{g_0^2}{2\mu_0^2} \left(\bar{\psi}\psi\right)^2 \tag{18.24}$$

∇_μ is a covariant derivative.

The first term of (18.24) describes kinetics of the fermi field and its interaction with the gravitational field, and the second one describes the interaction of the fields.

The effective potential of the model is written from the action

$$\Gamma_{eff}[\sigma] = -\frac{i}{2} \ln Det[\nabla^2 + \frac{1}{4}R - (g_0\sigma)^2] - \frac{\mu_0^2}{2}(\sigma, \sigma) \tag{18.25}$$

After making ultraviolet regularization and renormalizations of the model we get

$$L_{tot} = L_g + L_m = (L_g + L_m(\infty)) + L_m(\beta)$$

$$= L_g^{ren} + L_m(\beta) \tag{18.26}$$

The renormalized gravitational constant G_R and the $\tilde{\Lambda}_R$-term are:

$$\frac{1}{8\pi G_R}\tilde{\Lambda}_R = \frac{1}{8\pi G_0}\tilde{\Lambda}_0 + \frac{1}{16\pi^2}\Lambda^2 - \frac{1}{16\pi^2} m^2 \ln \Lambda^2$$

$$\frac{1}{16\pi G_R} = \frac{1}{16\pi G_0} + \frac{1}{16\pi^2} \ln \Lambda^2 \tag{18.27}$$

where Λ is a cut-off parameter, and m is a dynamical mass.

In the calculations (18.26) and (18.27) there was used the cut-off method of regularization of divergent integrals [Bogoliubov & Shirkov 1980].

The effective potential can be written from (18.25) in the form

$$V_{eff}[\sigma] = \sum_{j=1}^{2} \hat{\alpha}_j(R) f^j(\beta g_0 \sigma) + \frac{\mu_0^2}{2}\sigma^2 \tag{18.28}$$

where

$$f^0(\beta g_0 \sigma) = \frac{2m^2}{\pi^2 \beta^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} K_{-2}(\beta g_0 \sigma)$$

$$= -\frac{4}{\beta} \int \frac{d^3 k}{(2\pi)^3} \ln\left(1 + e^{-\beta\varepsilon}\right) \tag{18.29}$$

and

$$f^j(\beta g_0 \sigma_0) = \frac{1}{4 g_0} \left(-\frac{\partial}{\partial \sigma^2}\right)^j f^0(\beta g_0 \sigma) \tag{18.30}$$

The coefficients $\hat{\alpha}_j(R)$ are

$$\hat{\alpha}_0 = 1, \quad \hat{\alpha}_0 = \frac{1}{12} R, \quad .... \tag{18.31}$$

The solution of the gap equation

$$\frac{\partial}{\partial\sigma}V_{eff}[\sigma]_{|\sigma=\sigma_c} = 0 \tag{18.32}$$

gives, in high temperature approximation (XXII.33), the expression for dynamical mass without redefinition of the coupling constant and temperature

$$m^2(R,T) = (g_0\sigma_c)^2 = A/\lambda + b\cdot T^2 + C\cdot R + ... \tag{18.33}$$

where the constants

$$A = 32\pi^2/3.84, \quad B = A/24, \quad C = 0.7/12, \quad \lambda = (g_0/\mu_0)^2$$

are numerical positive coefficients. As we can see from the equation (18.33), the effective dynamical mass of fernion is a positive function for any temperature and curvature.

PART IV

TOPOLOGICALLY MASSIVE GAUGE THEORIES

Introduction

3-D non-Abelian gauge models describing systems of point particles carrying non-Abelian charge have been under investigation for over two decades. These models (so called topologically massive models) have a number of interesting features:

1) For vector fields, these models possess single, parity-violating, massive, spin 1 excitations, in contrast to single, massless, spin 0 exitations in the Maxwell theory, and to a pair of spin 1 degrees of freedom in gauge non-invariant models with a mass [Binegar 1982]

2) For second-rank tensor fields, describing gravity, the topological model leads to a single, parity-violating, spin 2 particle, whereas a conventional (gauge non-invariant) mass term gives rise to a spin 2 doublet. Furthermore, the topological term is of third-derivative order, yet the single propagating mode is governed by the Klein-Gordon equation. Einstein gravity, which is trivial and without propagation in three dimensions, becomes a dynamical theory with propagating particles.

3) Particles interacting via the Abelian Cherm-Simons term (CS-term) acquire anomalous spin and fractional statistics. They are called anyons [Wilczek 1977]. Anyons play a role in the fractional quantum Hall effect [Arovas et al. 1985], [Lee & Fisher 1989] and perhaps also in high temperature superconductivity [Lauglin 1988].

This is not a complete list of interesting properties of gauge theories in an odd number of dimensions. An important part of these models is the Chern-Simons action (CS-term). Some interesting aspects of quantum field models arising from the topology of odd-dimensional manifolds are discussed in chapter XIX. The origin of the CS-term of vector type that is induced by gauge interaction of 3-D fermions is studied in chapter XX. In this chapter the influence of topology and temperature effects are also considered. The origin of the induced gravitational CS-term at finite temperature is considered in chapter XXI.

Chapter 19

INTRODUCTION

TO TOPOLOGICAL FIELD MODELS

As an introduction to odd dimensional topological field models let us consider their origin and topological significance.

Topological aspect of the model

From gauge-invariant fields in even dimensions we may construct gauge invariant Pontryagin densities:

$$P_{(2)} = -(1/2\pi)\epsilon^{\mu\nu *}F^{\mu\nu}$$

$$P_{(4)} = -(1/16\pi^2)\mathrm{tr}^* F^{\mu\nu} F_{\mu\nu} \tag{19.1}$$

whose integrals over the even dimensional space are invariants that measure the topological content of the model.

These gauge invariant objects can also be written as total derivatives of gauge invariant quantities

$$P_n = \partial_\mu X_n^\mu \tag{19.2}$$

The two and four-dimensional expressions are

$$X_2^\mu = (1/2\pi)\epsilon^{\mu\nu} A_\nu$$

$$X_4^\mu = (1/2\pi)\epsilon^{\mu\alpha\nu\beta}\mathrm{tr}(A_\alpha F_{\beta\gamma} - (2/3)A_\alpha A_\beta A_\gamma) \tag{19.3}$$

The Chern-Simons (CS) secondary characteristic class is obtained by integrating one component of X_n^μ over the $(n-1)$ dimensional space which does not include that component.

Therefore the 3-D action S_{CS} is proportional to

$$S_{CS} \sim \int dx^0 dx^1 dx^2 X_4^3 \tag{19.4}$$

A topological massive term (we will name it CS term) can be added to the fundamental action for a gauge fields, but unlike the ways in which gauge fields are usually given a mass, no gauge symmetry is broken by its introduction.

Quantum aspects of 3-D field theory

Let us consider non-Abelian quantum model with topological mass term. The Lagrangian of this model is

$$L = L_0 + L_{CS} + L_{gauge} \tag{19.5}$$

where L_0 is the usual action for non-Abelian gauge field

$$L_0 = -(1/2)\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) \tag{19.6}$$

with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g[A_\mu, A_\nu] \tag{19.7}$$

L_{CS} is CS term

$$L_{CS} = -im\epsilon^{\mu\nu\rho}\mathrm{tr}(A_\mu \partial_\nu A_\rho - (2/3)A_\mu A_\nu A_\rho) \tag{19.8}$$

and L_{gauge} includes the gauge-fixing term

$$L_{gauge} = (\partial_\mu \bar{\eta}^a)(\partial^\mu \eta^a) + gf_{abc}(\partial_\mu \bar{\eta}^a)D^\mu \eta \tag{19.9}$$

We introduce $SU(N)$ gauge group here with matrix notation: $A_\mu = A^a_\mu \tau^a$, where τ^a

are anti-Hermitian matrices in the fundamental representation:

$$[\tau^a, \tau^b] = f^{abc}\tau^c, \quad \mathrm{tr}(\tau^a\tau^b) = -(1/2)\delta^{ab} \tag{19.10}$$

and f^{abc} are the structure constants of $SU(N)$.

The theory is defined in three space-time dimensions with Euclidean signature $(+\ +\ +)$. The coupling of the CS term is imaginary in Euclidean space-time and real in Minkowski space-time.

Properties of the model:

For an odd number of dimensions, the operation of parity, P, can be defined as reflection in all axes:

$$x^\mu \underset{P}{\rightarrow} -x^\mu, \quad A_\mu \underset{P}{\rightarrow} -A_\mu \tag{19.11}$$

The usual gauge Lagrangian is even under parity,

$$L_0 \underset{P}{\rightarrow} +L_0 \tag{19.12}$$

but the CS term is odd

$$L_{CS} \underset{P}{\rightarrow} -L_{CS} \tag{19.13}$$

Under gauge transformation

$$A_\mu \to \Omega^{-1}\left\{(1/g)\partial_\mu + A_\mu\right\}\Omega \tag{19.14}$$

The Lagrangian L_0 is invariant but L_{CS} is not:

$$\int d^3x L_{CS} \to \int d^3x L_{CS}$$

$$+ (im/g)\int d^3x \epsilon^{\mu\nu\rho}\partial_\mu \mathrm{tr}[(\partial_\nu\Omega)\Omega^{-1}A_\rho] + 8\pi^2(m/g^2)i\omega \tag{19.15}$$

where:

$$\omega = (1/24)\int d^3x \epsilon^{\mu\nu\rho}\mathrm{tr}[\Omega^{-1}(\partial_\mu\Omega)\Omega^{-1}(\partial_\nu\Omega)\Omega^{-1}(\partial_\rho\Omega)] \tag{19.16}$$

The set of gauge transformations is divided into global gauge rotations, $\partial_\mu\Omega = 0$, and all others, for which we assume that $\Omega(x) \to 1$ as $x^\mu \to \infty$. Integrating over global gauge rotations requires the system to have a total color charge equal to zero. In this case, $A_\mu(x)$ falls off faster than $1/|x|$ as $x^\mu \to \infty$, and the second term on the right-hand side of (19.15), which is a surface integral, vanishes.

The last term in (19.15) does not vanish in general. The ω of (19.16) is a winding number, which labels the homotopy class of $\Omega(x)$ [Nakahara 1992]

Changing variables $A \to A^U$, where A^U is a gauge transformation of A, implies that the vacuum average of the value $\hat{Q}$ is

$$< Q >= \exp |i8\pi^2(m/g^2)\omega(U)| < Q > \tag{19.17}$$

This invariance gives us a quantization condition for the dimensionless ratio

$$4\pi^2(m/g^2) = n, \quad n = 0, \pm 1, \pm 2, \tag{19.18}$$

Therefore, for the theory to be invariant under certain large gauge transformations (for a non-Abelian gauge group), which are not continiously deformable to the identity, the ratio of the CS mass m and the gauge coupling g^2 must be quantized [Deser et al. 1982].

Further we can study the problem of massive excitations.

For spinor electrodynamics in three dimensions we have

$$L = L_g + L_f + L_{int} \tag{19.19}$$

where

$$L_g = -(1/4)F^{\mu\nu}F_{\mu\nu} + (\mu/4)\epsilon^{\mu\nu}F_{\mu\nu}A_\alpha, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{19.20}$$

$$L_f = i\bar{\psi}\hat{\partial}\psi - m\bar{\psi}\psi \tag{19.21}$$

$$L_{int} = -J^\mu A_\mu, \quad J^\mu = -e\bar{\psi}\gamma^\mu\psi \tag{19.22}$$

The coupling constant e has dimension $(mass)^{-1/2}$.

The equations of motion will be

$$\partial_\mu F^{\mu\nu} + (\mu/4)\epsilon^{\nu\mu\alpha}F_{\mu\alpha} = J^\mu \tag{19.23}$$

$$(i\bar{\partial} + e\bar{A} - m)\psi = 0 \tag{19.24}$$

One can introduce the dual field strength tensor in 3-D space-time

$$^*F^\mu = (1/2)\epsilon^{\mu\alpha\beta}F_{\alpha\beta} \quad F^{\alpha\beta} = \epsilon^{\alpha\beta\mu*}F_\mu \tag{19.25}$$

The Bianchi identity follows from (19.23):

$$\partial_\mu{}^*F^\mu = 0 \tag{19.26}$$

and the equation (19.23) may be written in a dual form

$$\partial_\alpha{}^*F_\beta - \partial_\beta{}^*F_\alpha - \mu F_{\alpha\beta} = -\epsilon_{\alpha\beta\mu}J^\mu \tag{19.27}$$

or

$$(\Box + \mu^2)^*F^\mu = \mu\left(\eta^{\mu\nu} - \epsilon^{\mu\nu\alpha}\frac{\partial_\alpha}{\mu}\right)J_\nu \tag{19.28}$$

This equation demonstrates that the gauge excitations are massive.

3-D gravity. Connection with topology

The results of the topological part of the introduction gives us a hint how to construct the topological term for three dimensional gravity from a four-dimensional $^*RR$ Pontryagin density.

$$^*RR = (1/2)\epsilon^{\mu\nu\alpha\beta}R_{\mu\nu\rho\sigma}R_{\alpha\beta}{}^{\rho\sigma} = \partial_\mu X^\mu \tag{19.29}$$

Let us find X^μ from (19.29). To do this we will rewrite $^*RR$ in the following way

$$^*RR = (1/2)\epsilon^{\mu\nu\alpha\beta}R_{\mu\nu\rho\sigma}R_{\alpha\beta}{}^{\rho\sigma} = (1/2)\epsilon^{\mu\nu\alpha\beta}R_{\mu\nu ab}R_{\alpha\beta}{}^{ab}$$

$$= (1/2)\epsilon^{\mu\nu\alpha\beta}R_{\mu\nu ab}\left\{\partial_\alpha\omega_\beta{}^{ab} - \partial_\beta\omega_\alpha{}^{ab} + \omega_\alpha{}^{ac}\omega_{\beta c}{}^b - \omega_\beta{}^{ac}\omega_{\alpha c}{}^b\right\}$$

$$= \epsilon^{\mu\nu\alpha\beta}R_{\mu\nu ab}\left\{\partial_\alpha\omega_\beta{}^{ab} + \omega_\alpha{}^{ac}\omega_{\beta c}{}^b\right\} \tag{19.30}$$

In our calculations we used the following expression for the Ricci connection

$$R_{\mu\nu\rho\sigma} = \partial_\alpha\omega_{\beta ab} - \partial_\beta\omega_{\alpha ab} + \omega_{\alpha a}{}^c\omega_{\beta c_b} - \omega_{\beta a}{}^c\omega_{\alpha cb} \tag{19.31}$$

where $\omega_{\mu ab}$ is 3-D spin connection.

Then the expression for Pontryagin density will be

$$^*RR = \epsilon^{\mu\nu\alpha\beta}R_{\mu\nu ab}\ \partial_\alpha\omega_\beta{}^{ab} + \epsilon^{\mu\nu\alpha\beta}R_{\mu\nu ab}\ \omega_\alpha{}^{ac}\omega_{\beta c}{}^b \tag{19.32}$$

Let us find these two contributions separately.

The first contribution to (19.32) gives

$$\epsilon^{\mu\nu\alpha\beta}R_{\mu\nu ab}\ \partial_\alpha\omega_\beta{}^{ab} = \epsilon^{\mu\nu\alpha\beta}\partial_\alpha(R_{\mu\nu ab}\omega_\beta{}^{ab}) - \epsilon^{\mu\nu\alpha\beta}\omega_\beta{}^{ab}\partial_\alpha R_{\mu\nu ab}$$

$$= \epsilon^{\mu\nu\alpha\beta}\partial_\alpha(R_{\mu\nu ab}\omega_\beta{}^{ab})$$

$$- \epsilon^{\mu\nu\alpha\beta}\omega_\beta{}^{ab}\partial_\alpha\left\{\partial_\mu\omega_{\nu ab} - \partial_\nu\omega_{\mu ab} + \omega^c_{\mu a}\omega_{\nu cb}\omega^c_{\nu a}\omega_{\mu cb}\right\} \tag{19.33}$$

$$= \epsilon^{\mu\nu\alpha\beta}\partial_\alpha(R_{\mu\nu ab}\omega_\beta{}^{ab}) - 2\epsilon^{\mu\nu\alpha\beta}\omega_\beta{}^{ab}\partial_\alpha\left\{\omega^c_{\mu a}\omega_{\nu cb}\right\} \tag{19.34}$$

The second contribution to (19.32) will be

$$\epsilon^{\mu\nu\alpha\beta}R_{\mu\nu ab}\ \omega_\alpha{}^{ac}\omega_{\beta c}{}^b =$$

$$- \epsilon^{\mu\nu\alpha\beta}\left\{\partial_\mu\omega_{\nu ab} - \partial_\nu\omega_{\mu ab} + \omega^c_{\mu a}\omega_{\nu cb} - \omega^c_{\nu a}\omega_{\mu cb}\right\}\omega_\alpha{}^{ac}\omega_{\beta c}{}^b$$

$$= 2\epsilon^{\mu\nu\alpha\beta}\partial_\mu\omega_{\nu ab}\omega_\alpha{}^{ac}\omega_{\beta c}{}^b + 2\epsilon^{\mu\nu\alpha\beta}\omega^c_{\mu a}\omega_{\nu cb}\omega_\alpha{}^{ac}\omega_{\beta c}{}^b \tag{19.35}$$

On the other hand the second one to (19.35) is zero, then we have

$${}^*RR = \epsilon^{\mu\nu\alpha\beta}\partial_\alpha(R_{\mu\nu ab}\omega_\beta{}^{ab})$$

$$- 2\epsilon^{\mu\nu\alpha\beta}\omega_\beta{}^{ab}\partial_\alpha\left\{\omega^c_{\mu a}\omega_{\nu cb}\right\} + 2\epsilon^{\mu\nu\alpha\beta}\partial_\mu\omega_{\nu ab}\omega_\alpha{}^{ac}\omega_{\beta c}{}^b \tag{19.36}$$

One can find that

$$- 2\epsilon^{\mu\nu\alpha\beta}\omega_\beta{}^{ab}\partial_\alpha\left\{\omega^c_{\mu a}\omega_{\nu cb}\right\} + 2\epsilon^{\mu\nu\alpha\beta}\partial_\mu\omega_{\nu ab}\omega_\alpha{}^{ac}\omega_{\beta c}{}^b$$

$$= 2\epsilon^{\mu\nu\alpha\beta}\partial_\mu\omega^b_{\nu a}\omega_\alpha b^c\omega_{\beta c}{}^a \tag{19.37}$$

and

$$2\epsilon^{\mu\nu\alpha\beta}\partial_\mu\omega^b_{\nu a}\omega_\alpha b^c\omega_{\beta c}{}^a = (2/3)\epsilon^{\mu\nu\alpha\beta}\partial_\mu\left\{\omega^b_{\nu a}\omega_\alpha b^c\omega_{\beta c}{}^a\right\} \tag{19.38}$$

In the result we will have

$$^*RR = \partial_\mu\epsilon^{\mu\nu\alpha\beta}\left\{\omega_\nu{}^{ab}R_{\alpha\beta ab} + (2/3)\omega^b_{\nu a}\omega_{\alpha b}{}^c\omega_{\beta c}{}^a\right\} \tag{19.39}$$

Therefore

$$X^\mu = \epsilon^{\mu\nu\alpha\beta}\left\{\omega_\nu{}^{ab}R_{\alpha\beta ab} + (2/3)\omega_{\nu a}{}^b\omega_{\alpha b}{}^c\omega_{\beta c}{}^a\right\} \tag{19.40}$$

Let parameter μ be equal to zero, then determining $\epsilon^{0\nu\alpha\beta} = \epsilon^{\nu\alpha\beta}$ with $\nu, \alpha, \beta = 1, 2, 3$ we get the CS action in the form

$$S_{CS} \sim \int d^3xX^0 = \int d^3x\epsilon^{\nu\alpha\beta}\left\{\omega_\nu{}^{ab}R_{\alpha\beta ab} + (2/3)\omega_{\nu a}{}^b\omega_{\alpha b}{}^c\omega_{\beta c}{}^a\right\} \tag{19.41}$$

As we can see from (19.41) the CS term is of the third derivative order in contrast to the first one as in the vector case (19.3).

Nonlinear theory of gravity

We can construct total gravitational action in the form

$$S_{tot} = (1/k^2) \int d^3x \sqrt{g} R + (1/k^2 \mu) S_{CS} \tag{19.42}$$

The sign of the Einstein part is opposite to the conventional one in four dimensions. The Einstein part of the action has coefficient k^{-2} with the dimension of mass, while the topological part has a dimensionless coefficient (μ has a dimension of mass).

Now we can find some interesting properties from this action.

Varying (19.42) with respect to the metric, we get field equations

$$\Theta^{\mu\nu} \equiv G^{\mu\nu} + (1/\mu) C^{\mu\nu} = 0 \tag{19.43}$$

where the second rank Weyl tensor $C^{\mu\nu}$ is

$$C^{\mu\nu} = (1/\sqrt{g}) \epsilon^{\mu\alpha\beta} D_\alpha \tilde{R}^\nu_\beta \tag{19.44}$$

and

$$\tilde{R}_{\alpha\beta} = R_{\alpha\beta} - (1/4) g_{\alpha\beta} R, \quad R = R^\alpha_\alpha \tag{19.45}$$

The components of the Einstein tensor $G^{\alpha\beta}$ are

$$G^{\alpha\beta} = R^{\alpha\beta} - (1/2)g^{\alpha\beta}R, \tag{19.46}$$

and the components of the Riemann tensor $R^{\alpha}{}_{\beta\gamma\delta}$ are

$$R^{\alpha}{}_{\beta\gamma\delta} = \partial_\delta \Gamma^{\alpha}{}_{\beta\gamma} - \partial_\gamma \Gamma^{\alpha}{}_{\beta\delta} + \Gamma^{\alpha}{}_{\mu\gamma}\Gamma^{\mu}{}_{\beta\delta} - \Gamma^{\alpha}{}_{\mu\delta}\Gamma^{\mu}{}_{\beta\gamma} \tag{19.47}$$

From (19.43) we get first order form for field equations

$$K_{\mu\nu}{}^{\lambda\sigma}(\mu)R_{\lambda\sigma} = 0 \tag{19.48}$$

where $K_{\mu\nu}{}^{\lambda\sigma}$ is operator of the form

$$K_{\mu\nu}{}^{\lambda\sigma} = (\delta^{\lambda}_{\mu}\delta^{\lambda}_{\nu} - (1/2)g_{\mu\nu}g^{\lambda\sigma}) + \frac{1}{\mu\sqrt{g}}\epsilon_{\mu}{}^{\alpha\beta}(\delta^{\lambda}_{\beta}\delta^{\sigma}_{\nu} - (1/2)g^{\lambda\sigma}g^{\nu\beta}) \tag{19.49}$$

Operator $K_{\mu\nu}{}^{\lambda\sigma}(\mu)$ may be multiplied by $K_{\mu\nu}{}^{\lambda\sigma}(-\mu)$ to yield a second-order equation for Ricci tensor.

From (19.48) we find

$$K_{\alpha\beta}{}^{\mu\nu}(-\mu)K_{\mu\nu}{}^{\lambda\sigma}(\mu)R_{\lambda\sigma} = 0 \tag{19.50}$$

that is

$$(D_\alpha D^\alpha + \mu^2) R_{\mu\nu} = -g_{\mu\nu} R^{\alpha\beta} R_{\alpha\beta} + 3 R^\alpha_\beta R^\beta_\alpha \tag{19.51}$$

This exhibits the massive character of the excitations.

After this short review of topological field models we will consider 3-D fermionic models interacting with vector and with tensor fields. These models may give effective induced topological action of the CS type.

Chapter 20

INDUCED CHERN-SIMONS MASS TERM

IN TOPOLOGICALLY NON-TRIVIAL SPACE-TIME

In gauge theories with fermions, the a topological mass term is induced by fermionic interactions [Redlich 1984]. If the topology of space-time is not trivial, topological parameters will appear in the effective action and in the CS Lagrangian. In this chapter we show that the CS term depends on topological parameters in such a way that the topological gauge invariance of the total action can be maintained.

Let us consider the case of a massive fermion field interacting with external gauge field in 3-D space-time.

20.1 Euclidean space-time. Trivial topology

The Euclidean action of this quantum system is given by

$$S = \int d^3x\bar{\psi}(x)(i\hat{\partial}_x + m + g\hat{A}(x))\psi(x) \tag{20.1}$$

The corresponding generating functional is

$$Z = \int D\bar{\psi}(x)D\psi(x)exp\left[-\int d^3x\bar{\psi}(x)(i\hat{\partial}_x + m + g\hat{A}(x))\psi(x)\right] \tag{20.2}$$

Here $A_\mu(x) = A^a_\mu(x)T^a$, and T^a are generators of the gauge transformations. We will choose to work with two-component Dirac spinors.

Euclidean Dirac matrices have the algebra:

$$\{\gamma_\mu, \gamma_\nu\} = -\delta_{\mu\nu} \tag{20.3}$$

This equation is satisfied by the matrices

$$\gamma_1 = i\sigma_1, \quad \gamma_2 = i\sigma_2, \quad \gamma_3 = i\sigma_3 \tag{20.4}$$

where σ_i are the Pauli matrices.

Furthermore, they satisfy

$$\gamma_\mu\gamma_\nu = -\delta_{\mu\nu} - \epsilon_{\mu\nu\rho}\gamma_\rho,$$

$$\mathrm{Tr}(\gamma_\mu\gamma_\nu\gamma_\rho) = 2\epsilon_{\mu\nu\rho}$$

$$\mathrm{Tr}(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\lambda\gamma_\sigma) = -2(\delta_{\mu\nu}\epsilon_{\rho\lambda\sigma} + \delta_{\rho\lambda}\epsilon_{\mu\nu\sigma} - \delta_{\nu\sigma}\epsilon_{\rho\lambda\mu} + \delta_{\mu\sigma}\epsilon_{\rho\lambda\nu}) \tag{20.5}$$

The integration in (20.2) over the fermionic fields gives

$$Z[A_\mu] = Det(i\hat{\partial} + m + g\hat{A})) = exp(-S_{eff}) \tag{20.6}$$

Therefore we can write the expression for the effective action

$$S_{eff} = -\ln Det(i\hat{\partial} + m + g\hat{A})) = -\mathrm{Tr}\ln(i\hat{\partial} + m + g\hat{A}) \tag{20.7}$$

Now one can split (20.7) in two parts

$$S_{eff} = -\mathrm{Tr}\ln(i\hat{\partial}_x + m + g\hat{A}(x))$$

$$= \mathrm{Tr}\ln\hat{S}_f - \mathrm{Tr}\ln\left[1 + g\hat{S}_f\hat{A}\right] \tag{20.8}$$

where

$$\hat{S}_f = \frac{1}{i\hat{\partial} + m} = \frac{i\hat{\partial} + m}{\partial^2 + m^2} \tag{20.9}$$

is a fermionic propagator.

The first term of the equation (20.8) is a divergent gauge-independent contribution, and the second one may be expanded in a power series [Babu et al. 1987]:

$$S_{eff} = -\mathrm{Tr}\ln\left[1 + g\hat{S}_f\hat{A}\right] = \mathrm{Tr}\sum_{n=1}^{\infty}\frac{1}{n}[g\hat{S}_f\hat{A}] \tag{20.10}$$

Let us consider the term which is quadratic in $A_\mu(x)$. The corresponding action is given by

$$S^{(2)}_{eff} = \frac{1}{2}g^2\mathrm{Tr}\left[\frac{i\hat{\partial}+m}{\partial^2+m^2}\hat{A}\frac{i\hat{\partial}+m}{\partial^2+m^2}\hat{A}\right] \tag{20.11}$$

The terms with two and four γ matrices contribute to the wave function renormalization of the gauge boson and will not be taken into consideration in further calculations.

The other two terms are linear with respect to mass m and give

$$S^{(2)}_{eff} = \frac{1}{2}g^2 m\mathrm{Tr}\left[\frac{i\hat{\partial}}{\partial^2+m^2}\hat{A}\frac{1}{\partial^2+m^2}\hat{A} + \frac{1}{\partial^2+m^2}\hat{A}\frac{i\hat{\partial}}{\partial^2+m^2}\hat{A}\right]$$

$$= \frac{img^2}{2}\mathrm{tr}_D\gamma_\mu\gamma_\nu\gamma_\rho\mathrm{tr}\left[\frac{\partial_\mu}{\partial^2+m^2}A_\nu\frac{1}{\partial^2+m^2}A_\rho + \frac{1}{\partial^2+m^2}A_\mu\frac{\partial_\nu}{\partial^2+m^2}A_\rho\right] \tag{20.12}$$

where $\mathrm{Tr} = \mathrm{tr}_D\mathrm{tr}$ is the trace of both the γ matrices and the vector field, and also includes the integrations.

To write tr in the form $\mathrm{tr}\,[f(\partial)g(x)]$ one can use the relation

$$\phi(x)\frac{1}{\partial^2+m^2} = \frac{1}{\partial^2+m^2}\phi(x) + \frac{[\partial^2,\phi(x)]}{(\partial^2+m^2)^2} + \frac{[\partial^2,[\partial^2,\phi(x)]]}{(\partial^2+m^2)^3} + \ldots \tag{20.13}$$

Then

$$S_{eff}^{(2)} = img^2\ \epsilon_{\mu\nu\rho}\mathrm{Tr}\left[\frac{1}{(k^2+m^2)^2}(\partial_\nu A_\mu)A_\rho)\right]$$

$$= img^2\ \epsilon_{\mu\nu\rho}\int d^3x < x|\frac{1}{(\partial^2+m^2)^2}|x > tr\{(\partial_\nu A_\mu)A_\rho\} \tag{20.14}$$

In another form

$$S_{eff}^{(2)} = -img^2\ \epsilon_{\mu\nu\rho}\int d^3x\left[\int\frac{d^3k}{(2\pi)^3}\frac{1}{(k^2+m^2)^2}\right]\{(\partial_\mu A_\nu)A_\rho\} \tag{20.15}$$

The additional contribution to the CS term comes from the terms which are cubic in A_μ's. We can rewrite them as

$$S_{eff}^{(3)} = \frac{1}{3}g^3 m\mathrm{Tr}\left[\frac{i\hat{\partial}+m}{\partial^2+m^2}\hat{A}\frac{i\hat{\partial}+m}{k^2+m^2}\hat{A}\frac{i\hat{\partial}+m}{\partial^2+m^2}\hat{A}\right] \tag{20.16}$$

Noticing that the terms having even number of γ matrices do not contribute to the CS term, we have:

$$S_{eff}^{(3)} = -\frac{1}{3}g^3 mTr\left[\frac{\hat{\partial}}{\partial^2+m^2}\hat{A}\frac{\hat{\partial}}{\partial^2+m^2}\hat{A}\frac{1}{\partial^2+m^2}\hat{A}\right.$$

$$+\frac{\hat{\partial}}{\partial^2+m^2}\hat{A}\frac{1}{\partial^2+m^2}\hat{A}\frac{\hat{\partial}}{\partial^2+m^2}\hat{A}$$

$$+\frac{1}{\partial^2+m^2}\hat{A}\frac{\hat{\partial}}{\partial^2+m^2}\hat{A}\frac{\hat{\partial}}{\partial^2+m^2}\hat{A}$$

$$-m^2\frac{1}{\partial^2+m^2}\hat{A}\frac{1}{\partial^2+m^2}\hat{A}\frac{1}{\partial^2+m^2}\hat{A}\Big] \qquad (20.17)$$

Using the identity (20.13) find

$$S_{eff}^{(3)} = -\frac{mg^3}{3}\Big[\mathrm{tr}_D\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\lambda\gamma_\sigma \times$$

$$\times\,\mathrm{tr}\left\{\frac{\partial_\mu\partial_\rho}{(\partial^2+m^2)^3}A_\nu A_\lambda A_\sigma + \frac{\partial_\mu\partial_\lambda}{(\partial^2+m^2)^3}A_\nu A_\rho A_\sigma + \frac{\partial_\nu\partial_\lambda}{(\partial^2+m^2)^3}A_\mu A_\rho A_\sigma\right\}$$

$$-m^2\mathrm{tr}_D\gamma_\mu\gamma_\nu\gamma_\rho\mathrm{tr}\frac{1}{(\partial^2+m^2)^3}A_\mu A_\nu A_\rho\Big] \qquad (20.18)$$

Tracing with respect to γ matrices, find

$$S_{eff}^{(3)} = -\frac{2mg^3}{3}\left[\int\frac{d^3k}{(2\pi)^3}\frac{1}{(k^2+m^2)^2}\right]\mathrm{tr}\int d^3x\epsilon^{\mu\nu\rho}A_\mu A_\nu A_\rho \qquad (20.19)$$

Combining (20.15) with (20.19) we get [Babu et al. 1987] the induced Chern-Simons term in the form

$$S_{eff}^{CS} = -img^2\left[\int\frac{d^3k}{(2\pi)^3}\frac{1}{(k^2+m^2)^2}\right]\mathrm{tr}\int d^3x\epsilon_{\mu\nu\rho}\left[\partial_\mu A_\nu A_\rho - \frac{2}{3}igA_\mu A_\nu A_\rho\right] \qquad (20.20)$$

or, after integration over the momenta

$$S_{eff}^{CS} = -\frac{ig^2}{8\pi}\frac{m}{|m|}\mathrm{tr}\int d^3x\epsilon_{\mu\nu\rho}\left[\partial_\mu A_\nu A_\rho - \frac{2}{3}igA_\mu A_\nu A_\rho\right] \qquad (20.21)$$

Now we can apply the method we have developed for a gauge model in 3-D non-trivial space-time.

20.2 Non-trivial topology

a) Model with space-time topology $\Sigma = R^{(2)} \times mobius\ strip$.

To introduce topology $\Sigma = R^{(2)} \times S^1$ with parameter ζ we rewrite the expression (20.15) in the following way. Momentum integration in (20.15) will be integration with respect to two dimensions $((t,x) \to R^{(2)})$,and the third integration will be transformed in the sum, because the fermionic propagator is antisymmetric with respect to the selected axis $(y \to S^1)$.

We can consider that $k^2 = \vec{k}^2 + \omega_n^2$ with $\omega_n = (2\pi/\zeta)(n+1/2)$ and $n = 0, \pm 1, \pm 2, ...$ Integrating in 2-D momentum space we find that

$$\tilde{\int} \frac{d^3k}{(2\pi)^3} \frac{1}{(k^2+m^2)^2} = \int \frac{d^2k}{(2\pi)^2} \frac{1}{\zeta} \sum_{n=-\infty}^{\infty} \frac{1}{((\vec{k})^2 + \omega_n^2 + m^2)^2}$$

$$= \frac{1}{8\pi|m|} tanh \left\{ \frac{|m|\zeta}{2} \right\} \tag{20.22}$$

where

$$\tilde{\int} \frac{d^3k}{(2\pi)^3} = \frac{1}{\zeta} \sum_{n=-\infty}^{\infty} \int \frac{d^2k}{(2\pi)^2}$$

In the process of calculations we used the useful equation

$$\sum_{n=-\infty}^{\infty} \frac{y}{(y^2 + (n+1/2)^2)} = \pi tanh(\pi y) \tag{20.23}$$

Then the expression (20.20) in this topology will be

$$S_{eff}^{(CS)} = -img^2 \int \frac{d^2k}{(2\pi)^2} \frac{1}{\zeta} \sum_{n=-\infty}^{\infty} \frac{1}{((\vec{k})^2 + \omega_n^2 + m^2)^2}$$

$$\mathrm{tr} \int \epsilon_{\mu\nu\rho} \left\{ \partial_\nu A_\mu A_\rho - \frac{2i}{3} g A_\nu A_\mu A_\rho \right\} \qquad (20.24)$$

and the induced CS term at non-trivial 3-D space-time is written as

$$S_{eff}^{CS}(\zeta) = -\frac{ig^2}{8\pi} \frac{m}{|m|} tanh \left\{ \frac{|m|\zeta}{2} \right\} \mathrm{tr} \int d^3x \epsilon_{\mu\nu\rho} (\partial_\mu A_\nu A_\rho - \frac{2}{3} ig A_\mu A_\nu A_\rho) \qquad (20.25)$$

The relation between CS terms is the function of the form

$$\frac{S_{eff}^{CS}(\zeta)}{S_{eff}^{CS}} = tanh \left\{ \frac{|m|\zeta}{2} \right\} \qquad (20.26)$$

b) Model with space-time topology $\Sigma = R^{(2)} \times S^1$.

For this space-time topology the propagator of the fermionic field will be periodic at the boundary points of interval $[0, \zeta]$, that leads to the modification of equation (20.21) with respect to new frequencies $\omega_n = 2\pi n/\zeta$ with $n = 0, \pm 1, \pm 2, ...$

Taking into account the summation formula

$$\sum_{n=-\infty}^{\infty} \frac{y}{y^2 + n^2} = \pi coth(\pi y) \qquad (20.27)$$

we find, that

$$\int_{\Sigma} \frac{d^3k}{(2\pi)^3} \frac{1}{(k^2+m^2)^2} = \int \frac{d^2k}{(2\pi)^2} \frac{1}{\zeta} \sum_{n=-\infty}^{\infty} \frac{1}{((\vec{k})^2 + \omega_n^2 + m^2)^2}$$

$$= \frac{1}{8\pi|m|} coth \left\{ \frac{|m|\zeta}{2} \right\} \tag{20.28}$$

Using this equation and the equation (20.19), get the induced CS term in the new topology

$$S_{eff}^{CS}(\zeta) = -\frac{ig^2}{8\pi} \frac{m}{|m|} coth \left\{ \frac{|m|\zeta}{2} \right\} \operatorname{tr} \int d^3x \epsilon_{\mu\nu\rho} (\partial_\mu A_\nu A_\rho - \frac{2}{3} ig A_\mu A_\nu A_\rho) \tag{20.29}$$

The rotio of the CS terms will be the function of the topological parameter ζ:

$$\frac{S_{eff}^{CS}(\zeta)}{S_{eff}^{CS}} = coth \left\{ \frac{|m|\zeta}{2} \right\} \tag{20.30}$$

These results show that the relations (20.26) and (20.30) are smooth functions of the topological parameter ζ.

Chapter 21

GRAVITATIONAL

CHERN-SIMONS

MASS TERM

AT FINITE TEMPERATURE

In this chapter we will consider fermions interacting with an external gravitational field at finite temperature. In the previous chapter XX we learned that the interaction of fermions with external gauge bosons leads to the induced action of the Chern-Simons type. In the same way the interaction of fermions with external gravitational fields may leads to the effective gravitational Chern-Simons term [Redlich 1984], [Alvarez-Gaume et al. 1985]. Now we will develop a formalism of calculations of the effective gravitational CS action from the action for massive fermions interacting with an external gravitational field.

21.1 Induced gravitational Chern-Simons mass term

Let us introduce the action for massive fermions connected with an external gravitational field as:

$$S = \int d^3x \sqrt{g}\bar{\psi}(x)(iD + m)\psi(x) \tag{21.1}$$

where $D = \gamma^\mu D_\mu = \gamma^\mu(\partial_\mu + \omega_{ab\mu}\sigma^{ab})$ with $\gamma^\mu(x) = h^\mu_a(x)\gamma^a$.

The function $\omega_{ab\mu}$ is the local Lorentz connection and $\sigma^{ab} = (1/8)[\gamma^a, \gamma^b]$ is the commutator of γ matrices.

γ-matrices with Latin indices are constructed from Pauli matrices

$$\gamma^a = i\sigma^a \quad , \sigma^a(a = 1, 2, 3) \tag{21.2}$$

and obey the relations:

$$\{\gamma^a, \gamma^b\} = -2\delta^{ab}, \quad \gamma_a\gamma_b = -\delta_{ab} - \epsilon_{abc}\gamma^c \tag{21.3}$$

$$\mathrm{tr}(\gamma^a\gamma^b) = -2i\delta^{ab}, \quad \mathrm{tr}(\gamma^a\gamma^b\gamma^c) = 2\epsilon^{abc} \tag{21.4}$$

We can find the effective action by integrating (21.1) with respect to the fermionic field from the equation

$$exp[-S_{eff}] = \int D\bar{\psi}(x)\psi(x)\exp[-S] = Det(-iD - m) \tag{21.5}$$

To find the one-loop effective action we will rewrite (21.5) in the following form [Ojima 1989]:

$$S_{eff} = -ln\ Det\frac{(-iD - m)}{(-iD + i\omega - m)}$$

$$= -\ln\ Det\frac{(-iD - m)(-iD - i\omega - m)}{(-iD + i\omega - m)(-iD - i\omega - m)}$$

$$= -\ln\ Det(DD - D\omega + im\omega + m^2) + logDet[(D - \omega)^2] + m^2 \tag{21.6}$$

where we put $\omega = \gamma^\mu \omega_\mu = \gamma^\mu \sigma^{ab} \omega_{ab\mu}$.

Now we can use the momentum space formalism we developed in part I of this work for calculation of the first determinant of the expression (21.6). We can simplify the problem by noticing that the second term of (21.6) does not give contributions to the induced CS term.

For the calculation of the first term let us introduce normal coordinates and write the metric at the origin (x') of these coordinates. We will use variable $y = x - x'$ for the definition of the point of manifold in tangent space. The tangent space Y will be "flat" with metric $g_{ab} = -\eta_{ab} = diag(-1,-1,-1)$. In the origin of these coordinates the tetrad functions and the metric will be the series:

$$h_a^\mu(x) = h_a^\mu(x^{'}) - (1/6)R^\mu{}_{\nu a\sigma}y^\mu y^\sigma + ... \tag{21.7}$$

$$g_{\mu\nu}(x) = g_{\mu\nu}(x^{'}) - (1/3)R_{\mu\alpha\nu\beta}y^\mu y^\nu - (1/6)R_{\mu\alpha\nu\beta;\lambda}y^\mu y^\nu y^\lambda + ... \tag{21.8}$$

The covariant derivative in a tangent space is written as

$$D = \gamma_\mu(x)D_\mu(x)$$

$$= \left[\gamma^a h_a^\mu(x^{'}) - (1/6)\gamma^a R^\mu{}_{\nu a\sigma}y^\mu y^\sigma + ...\right]\left(\frac{\partial}{\partial y^\mu} + \omega_{ab\mu}(x^{'})\sigma^{ab}\right) \tag{21.9}$$

The effective action can be represented in the normal coordinates as

$$S_{eff} = \int d^3x\sqrt{g}S_{eff}(x)$$

$$= \lim_{x\to x^{'}}\int d^3x^{'}\sqrt{g(x^{'})}S_{eff}(x^{'}) = \lim_{y\to 0}\int d^3x\sqrt{g}S_{eff}(x,y) \tag{21.10}$$

In proper time formalism $lnDet$ is

$$S_{eff}[\omega] = \ln\ Det(DD - D\omega + im\omega + m^2) =$$

$$\lim_{x\to x^{'}}\int d^3x < x|\int_0^\infty \frac{ds}{s}\mathrm{tr}\exp\left\{-i\left(D^\mu D_\mu + \frac{R}{4} - D\omega + im\omega + m^2\right)\right\}s|x^{'} > \tag{21.11}$$

In accordance with (21.10) and (21.11) the density $S_{eff}(x)$ may be written in the momentum-space form

$$S_{eff}(x) = \lim_{y\to 0} \int_0^\infty \frac{ds}{s} \int \frac{d^3k}{(2\pi)^3} \exp[\ iky\ ] \exp[-i(m^2 - \Delta_\mu \Delta^\mu)s] \times$$

$$\operatorname{tr} \exp\left\{-i\left[\left(D^\mu D_\mu + \frac{R}{4} - D\omega\right) + (i\Delta_\mu D^\mu + iD_\mu \Delta^\mu - i\Delta\omega + im\omega)\right] s\right\} \quad (21.12)$$

where $\lim\limits_{y\to 0} \Delta_\mu(y) = k_\mu$.

Notice that if we like to obtain the contribution of finite terms as $(m \to 0)$, it is sufficient to extract the terms which are proportional to s^2 with a factor m or proportional to s^3 with a factor m^3 (or k^2). Thus we get the term in $S_{eff}[\omega]$, which contains the induced CS term.

The second and the third orders of the expansion (21.12) give

$$S_{eff}(x) = \lim_{y\to 0} \int \frac{d^3k}{(2\pi)^3} \exp[\ iky\ ] \int_0^\infty \frac{ds}{s} \exp[-i(m^2 - \Delta_\mu \Delta^\mu)s](im) \times$$

$$\times \operatorname{tr} \left[\frac{1}{2!} \left(\omega D\omega + D\omega\omega\right) s^2 - \right.$$

$$\left. -\frac{1}{3!}(\Delta\omega\Delta\omega\omega + \Delta\omega\omega\Delta\omega + \omega\Delta\omega\Delta\omega + m^2\omega\omega\omega)s^3\right] \quad (21.13)$$

where $\Delta(y) = \gamma^\mu \Delta_\mu(y)$.

In the limit $(y \to 0)$ the equation (21.13) is

$$S_{eff}(x) = (im) \int \frac{d^3k}{(2\pi)^3} \int_0^\infty \frac{ds}{s} \exp[-i(m^2 + k^2)s] \times$$

$$\times \text{ tr}\left[\left(\omega\hat{\partial}\omega + \omega\omega\omega\right)\ s^2 -\right.$$

$$\left. -\frac{1}{3!}(\hat{k}\omega\hat{k}\omega\omega + \hat{k}\omega\omega\hat{k}\omega + \omega\hat{k}\omega\hat{k}\omega + m^2\omega\omega\omega)s^3\right] \tag{21.14}$$

Noticing, that

$$\text{tr}(\hat{k}\omega\hat{k}\omega\omega + \hat{k}\omega\omega\hat{k}\omega + \omega\hat{k}\omega\hat{k}\omega + m^2\omega\omega\omega) = (k^2 + m^2)\text{tr}\omega\omega\omega \tag{21.15}$$

rewrite (21.14) as

$$S_{eff}(x) = (im)\left[\text{tr}\omega\hat{\partial}\omega + \frac{2}{3}\text{tr}\omega\omega\omega\right]\int\frac{d^3k}{(2\pi)^3}\int\limits_0^\infty ds(s)\exp[-is(m^2 + k^2)] \tag{21.16}$$

Applying the equation for the $\Gamma(\nu)$ function

$$\int\limits_0^\infty dss^{\nu-1}\exp[-sm^2] = \frac{\Gamma(\nu)}{|m|^{2\nu}} \tag{21.17}$$

and computing tr of γ matrices, find

$$S^{CS}_{eff}[\omega] = -\frac{i}{64\pi}\frac{m}{|m|}\int d^3x\epsilon^{\mu\nu\rho}\left(\partial_\mu\omega^b{}_{a\nu}\omega^a{}_{b\rho} + \frac{2}{3}\omega^a{}_{b\nu}\omega^b{}_{c\mu}\omega^c{}_{a\rho}\right) \tag{21.18}$$

This is the expression for gravitational CS term [Ojima 1989] which we get using momentum-space method.

21.2 Induced gravitational Chern-Simons mass term at finite temperature

The calculations of finite temperature gravitational action of CS type are based on the computation of $\ln Det$ (21.5). In Part I we developed the formalism of such calculations with the help of the momentum space methods. For these calculations we will replace the integration procedure over three momenta in the expression (21.16) by the summation over Matsubara frequencies and integration over the two remaining momenta.

Let us introduce temperature in the tangent 3-D space to the curved manifold as (part I,(2.47)):

$$\int \frac{d^3k}{(2\pi)^3} F(k_1^2, k_2^2, k_3^2, s) \xrightarrow{T \neq 0} \frac{1}{\beta} \sum_{n=-\infty}^{\infty} \int \frac{d^2k}{(2\pi)^2} F(\omega_n^2, k_2^2, k_3^2, s) \tag{21.19}$$

where $\beta^{-1} = T$ is temperature, and $\omega_n^2 = (2\pi/\beta)(n + 1/2)$.

To introduce temperature in the model, rewrite integral over (s) in (21.16) in the form

$$\int \frac{d^3p}{(2\pi)^3} \int ds \; s \exp[-is(k^2 + m^2)] = -\int \frac{d^3p}{(2\pi)^3} \frac{1}{(k^2 + m^2)^2} \tag{21.20}$$

Using (21.19) and the summation formula (20.23), find the finite temperature CS term

$$S_{eff}[\omega] = -\frac{i}{64\pi} \frac{m}{|m|} tanh \frac{|m|}{2T} \int d^3x \epsilon^{\mu\nu\rho} \left(\partial_\mu \omega^b{}_{a\nu} \omega^a{}_{b\rho} + \frac{2}{3} \omega^a{}_{b\nu} \omega^b{}_{c\mu} \omega^c{}_{a\rho} \right) \tag{21.21}$$

Comparing (21.18) and (21.21) we get the following result

$$\frac{S_{eff}[T \neq 0]}{S_{eff}[T = 0]} = tanh \frac{\beta |m|}{2} \tag{21.22}$$

The expression (21.21) shows that the structure of the induced CS term at finite temperature is exactly the same as at zero temperature, and the functional relations between the CS terms in tensor (21.22) and vector types of interactions (20.26) are the same.

APPENDIX

SUMS OF BESSEL FUNCTIONS

I. Type I sums of the modified Bessel functions

The integral representations for series of modified Bessel functions may be calculated from the following summation formula:

$$\sum_{n=-\infty}^{\infty} \left[y^2 + \left(\frac{n+1}{2} \right)^2 \right]^{-1} = \frac{\pi}{y} \tanh(\pi y). \tag{XXII.1}$$

In proper time representation the left side of (XXII.1) can be written in the form

$$\sum_{n=-\infty}^{\infty} \left[y^2 + \left(\frac{n+1}{2} \right)^2 \right]^{-1}$$

$$= \int_0^{\infty} d\alpha \exp\left(-\alpha y^2\right) \sum_{n=-\infty}^{\infty} \exp\left(-\alpha \left(n + \frac{1}{2}\right)^2\right). \tag{XXII.2}$$

Taking into account the equation

$$\sum_{n=-\infty}^{\infty} \exp\left\{-\alpha \left(n - z\right)^2\right\} = \sum_{n=-\infty}^{\infty} \left(\frac{\pi}{\alpha}\right)^{\frac{1}{2}} \exp\left(-\frac{\pi^2}{\alpha} n^2 - 2\pi i z n\right), \tag{XXII.3}$$

we can write (XXII.2) as

$$\sum_{n=-\infty}^{\infty}\left[y^2+\left(\frac{n+1}{2}\right)^2\right]^{-1}=\sum_{n=-\infty}^{\infty}(-1)^n\int\limits_0^{\infty}d\alpha\left(\frac{\pi}{\alpha}\right)^{\frac{1}{2}}\exp\left(-\alpha y^2-\frac{\pi^2}{\alpha}n^2\right)$$

$$=\frac{\pi}{y}+2\sum_{n=1}^{\infty}(-1)^n\int\limits_0^{\infty}d\alpha\left(\frac{\pi}{\alpha}\right)^{\frac{1}{2}}\exp\left(-\alpha y^2-\frac{\pi^2}{\alpha}n^2\right). \qquad \text{(XXII.4)}$$

The right side of (XXII.1) is

$$\frac{\pi}{y}\tanh(\pi y)=\frac{\pi}{y}-\frac{2\pi}{y\left(e^{2\pi y}+1\right)}. \qquad \text{(XXII.5)}$$

Therefore, we get from (XXII.4) and (XXII.5) the following useful equation

$$\frac{1}{z\left(e^z+1\right)}=-\frac{1}{2\pi^2}\sum_{n=1}^{\infty}(-1)^n\int\limits_0^{\infty}d\alpha\left(\frac{\pi}{\alpha}\right)^{\frac{1}{2}}\exp\left(-\alpha\frac{z^2}{4\pi^2}-\frac{\pi^2}{\alpha}n^2\right). \qquad \text{(XXII.6)}$$

Moreover, we may consider that $z^2=g_a(x^2)$ is the function of variable $x\in R^3$ with a parameter a, namely $z^2=x^2+a^2$.

Integrating with respect to x one can get

$$\int\frac{d^3x}{(2\pi)^3}\left[\sqrt{x^2+a^2}\left(\exp\left(\sqrt{x^2+a^2}\right)+1\right)\right]^{-1}$$

$$=-\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n\int\limits_0^{\infty}d\alpha\alpha^{-2}\exp\left(-\alpha\frac{a^2}{4\pi^2}-\frac{\pi^2}{\alpha}n^2\right). \qquad \text{(XXII.7)}$$

The modified Bessel function may be written as

$$\int_0^\infty d\alpha \cdot \alpha^{\nu-1} \cdot \exp\left(-\gamma\alpha - \frac{\delta}{\alpha}\right) = 2\left(\frac{\delta}{\gamma}\right)^{\frac{\nu}{2}} K_\nu\left(2\sqrt{\delta\,\gamma}\right). \qquad \text{(XXII.8)}$$

Then (XXII.7) will be

$$\int \frac{d^3x}{(2\pi)^3}\left[\sqrt{x^2+a^2}\left(\exp\left(\sqrt{x^2+a^2}\right)+1\right)\right]^{-1}$$

$$= -\frac{1}{2}\left(\frac{a}{\pi^2}\right)\sum_{n=1}^{\infty}\frac{(-1)^n}{n}K_1(an). \qquad \text{(XXII.9)}$$

Scaling x and a with a parameter β as $(x,a) = \beta(k,m)$ write (XXII.9) in the form

$$\int \frac{d^3k}{(2\pi)^3}\frac{2}{\varepsilon\left(\exp\left(\beta\varepsilon\right)+1\right)} = -\frac{m}{\beta\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}K_1(\beta mn), \qquad \text{(XXII.10)}$$

where $\varepsilon = \sqrt{\vec{k}^2+m^2}$.

Differentiating the equation (XXII.7) with respect to parameter a we get

$$\int \frac{d^3x}{(2\pi)^3}\left(-\frac{\partial}{\partial a^2}\right)\left[\sqrt{x^2+a^2}\left(\exp\left(\sqrt{x^2+a^2}\right)+1\right)\right]^{-1}$$

$$= -\frac{1}{4\pi^2}\sum_{n=1}^{\infty}(-1)^n K_0(an), \qquad \text{(XXII.11)}$$

or, in new variables,

$$\int \frac{d^3k}{(2\pi)^3}\left(\frac{\partial}{\partial m^2}\right)\frac{1}{\varepsilon\left(\exp\left(\beta\varepsilon\right)+1\right)} = \frac{1}{4\pi^2}\sum_{n=1}^{\infty}(-1)^n K_0(\beta mn). \qquad \text{(XXII.12)}$$

Integrating in (XXII.7) with respect to parameter (a) and using the equation

$$\int\limits_{a^2}^{\infty} da^2 \left[\sqrt{x^2+a^2}\left(\exp\left(\sqrt{x^2+a^2}\right)+1\right)\right]^{-1}$$

$$= 2\ln\left(1+\exp\left(-\sqrt{x^2+a^2}\right)\right) \qquad \text{(XXII.13)}$$

we find with new variables the following equation

$$-\frac{1}{\beta}\int \frac{d^3k}{(2\pi)^3}\ln\left(1+\exp(-\beta\varepsilon)\right) = \frac{m^2}{2(\beta\pi)^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}K_2(\beta m n). \qquad \text{(XXII.14)}$$

High temperature asymptotes ($\beta m \ll 1$) of the equations (XXII.9), (XXII.11) and (XXII.14) are

$$\frac{2m^2}{(\beta\pi)^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}K_2(\beta m n) = -\frac{7\pi^2}{180\beta^4}+\frac{m^2}{12\beta^2}+\frac{1}{8}m^4\left(\ln\frac{\beta m}{4\pi}+\gamma-\frac{3}{4}\right) \text{(XXII.15)}$$

$$-\frac{m}{\beta\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}K_1(\beta m n) = \frac{1}{12\beta^2}+\frac{1}{4}m^2\left(\ln\left(\frac{\beta m}{4\pi}\right)+\gamma-\frac{1}{2}\right) \qquad \text{(XXII.16)}$$

$$\frac{1}{2\pi^2}\sum_{n=1}^{\infty}(-1)^n K_0(\beta m n) = \frac{1}{4}\left(\ln\left(\frac{\beta m}{4\pi}\right)+\gamma\right). \qquad \text{(XXII.17)}$$

II. Type II sums of the modified Bessel functions

Let us start with the sum:

$$\sum_{n=-\infty}^{\infty}\left(y^2+n^2\right)^{-1}=\frac{\pi}{y}\coth(\pi y). \tag{XXII.18}$$

The proper time representation the left side of (XXII.18) will be

$$\sum_{n=-\infty}^{\infty}\left(y^2+n^2\right)^{-1}=\int_0^\infty d\alpha \exp\left(-\alpha y^2\right)\sum_{n=-\infty}^{\infty}\exp\left(-\alpha n^2\right). \tag{XXII.19}$$

Taking into account the equation (XXII.3) and putting ($z=0$) we find

$$\sum_{n=-\infty}^{\infty}\exp\left\{-\alpha n^2\right\}=\sum_{n=-\infty}^{\infty}\left(\frac{\pi}{\alpha}\right)^{\frac{1}{2}}\exp\left(-\frac{\pi^2}{\alpha}n^2\right), \tag{XXII.20}$$

Then (XXII.19) may be written in the form

$$\sum_{n=-\infty}^{\infty}\left(y^2+n^2\right)^{-1}=\sum_{n=-\infty}^{\infty}\int_0^\infty d\alpha\left(\frac{\pi}{\alpha}\right)^{\frac{1}{2}}\exp\left(-\alpha y^2-\frac{\pi^2}{\alpha}n^2\right)$$

$$=\frac{\pi}{y}+2\sum_{n=1}^{\infty}\int_0^\infty d\alpha\left(\frac{\pi}{\alpha}\right)^{\frac{1}{2}}\exp\left(-\alpha y^2-\frac{\pi^2}{\alpha}n^2\right). \tag{XXII.21}$$

The right side of (XXII.18) is

$$\frac{\pi}{y}\coth(\pi y)=\frac{\pi}{y}+\frac{2\pi}{y\left(e^{2\pi y}-1\right)}. \tag{XXII.22}$$

Therefore, we get from (XXII.21) and (XXII.22) the following useful equation

$$\frac{1}{z\left(e^{z}-1\right)}=\frac{1}{2\pi^2}\sum_{n=1}^{\infty}\int_{0}^{\infty}d\alpha\left(\frac{\pi}{\alpha}\right)^{\frac{1}{2}}\exp\left(-\alpha\frac{z^2}{4\pi^2}-\frac{\pi^2}{\alpha}n^2\right). \tag{XXII.23}$$

Let $z^2 = g_a(x^2)$ be a function of variable $x \in R^3$ with a parameter a of the form $z^2 = x^2 + a^2$.

After integration with respect to x we get

$$\int\frac{d^3x}{(2\pi)^3}\left[\sqrt{x^2+a^2}\left(\exp\left(\sqrt{x^2+a^2}\right)-1\right)\right]^{-1} \tag{XXII.24}$$

$$=\frac{1}{2}\sum_{n=1}^{\infty}\int_{0}^{\infty}d\alpha\alpha^{-2}\exp\left(-\alpha\frac{a^2}{4\pi^2}-\frac{\pi^2}{\alpha}n^2\right). \tag{XXII.25}$$

The modified Bessel function is written as

$$\int_{0}^{\infty}d\alpha\cdot\alpha^{\nu-1}\cdot\exp\left(-\gamma\alpha-\frac{\delta}{\alpha}\right)=2\left(\frac{\delta}{\gamma}\right)^{\frac{\nu}{2}}K_{\nu}\left(2\sqrt{\delta\,\gamma}\right). \tag{XXII.26}$$

and (XXII.25) will be

$$\int\frac{d^3x}{(2\pi)^3}\left[\sqrt{x^2+a^2}\left(\exp\left(\sqrt{x^2+a^2}\right)-1\right)\right]^{-1}$$

$$=\frac{1}{2}\left(\frac{a}{\pi^2}\right)\sum_{n=1}^{\infty}\frac{1}{n}K_1(an). \tag{XXII.27}$$

Scaling x and (a) with a parameter β as $(x, a) = \beta(k, m)$ write (XXII.27) in the form

$$\int \frac{d^3k}{(2\pi)^3} \frac{2}{\varepsilon\left(\exp\left(\beta\varepsilon\right) - 1\right)} = \frac{m}{\beta\pi^2} \sum_{n=1}^{\infty} \frac{1}{n} K_1(\beta m n), \tag{XXII.28}$$

where $\varepsilon = \sqrt{\vec{k}^2 + m^2}$.

Differentiating the equation (XXII.25) with respect to parameter a we find

$$\int \frac{d^3x}{(2\pi)^3} \left(-\frac{\partial}{\partial a^2}\right) \left[\sqrt{x^2 + a^2}\left(\exp\left(\sqrt{x^2 + a^2}\right) - 1\right)\right]^{-1}$$

$$= \frac{1}{4\pi^2} \sum_{n=1}^{\infty} K_0(an), \tag{XXII.29}$$

or, in new variables,

$$\int \frac{d^3k}{(2\pi)^3} \left(\frac{\partial}{\partial m^2}\right) \frac{1}{\varepsilon\left(\exp\left(\beta\varepsilon\right) - 1\right)} = -\frac{1}{4\pi^2} \sum_{n=1}^{\infty} K_0(\beta m n). \tag{XXII.30}$$

Integrating (XXII.25) with respect to parameter (a) and using the equation

$$\int_{a^2}^{\infty} da^2 \left[\sqrt{x^2 + a^2}\left(\exp\left(\sqrt{x^2 + a^2}\right) - 1\right)\right]^{-1}$$

$$= 2\ln\left(1 - \exp\left(-\sqrt{x^2 + a^2}\right)\right) \tag{XXII.31}$$

we find with new variables the following equation

$$-\frac{1}{\beta} \int \frac{d^3k}{(2\pi)^3} \ln\left(1 - \exp(-\beta\varepsilon)\right) = \frac{m^2}{2(\beta\pi)^2} \sum_{n=1}^{\infty} \frac{1}{n^2} K_2(\beta m n). \tag{XXII.32}$$

High temperature asymptotes ($\beta m \ll 1$) of the equations (XXII.27), (XXII.29) and (XXII.32) are

$$\frac{2m^2}{(\beta\pi)^2}\sum_{n=1}^{\infty}\frac{1}{n^2}K_2(\beta mn)$$

$$= \frac{\pi^2}{90\beta^4} - \frac{m^3}{24\beta^2} + \frac{m^3}{12\pi\beta} + \frac{m^4}{64}\left[\ln\frac{m^2\beta^2}{16\pi^2} - \frac{3}{2} + 2\gamma\right]$$

$$\frac{m}{4\beta\pi^2}\sum_{n=1}^{\infty}\frac{1}{n}K_1(\beta mn), \qquad \text{(XXII.33)}$$

$$= \frac{1}{24\beta^2} - \frac{m}{8\pi\beta} - \frac{m^2}{32\pi^2}\left[\ln\frac{m^2\beta^2}{16\pi^2} - 1 + 2\gamma\right] \qquad \text{(XXII.34)}$$

and

$$\frac{1}{4\pi^2}\sum_{n=1}^{\infty}K_0(\beta mn)$$

$$= \frac{1}{16\pi m\beta} + \frac{1}{32\pi^2}\left[\ln\frac{m^2\beta^2}{16\pi^2} - \frac{1}{2} + 2\gamma\right] \qquad \text{(XXII.35)}$$

GRAPHICS

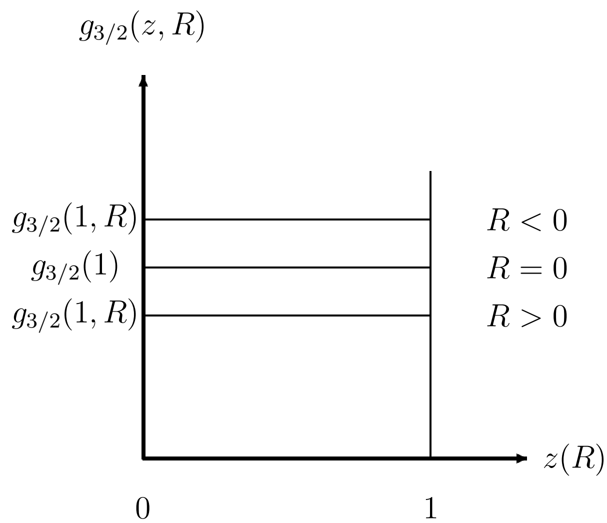


Fig. I-1 Graphical expression of the function $g_{3/2}(z, R)$

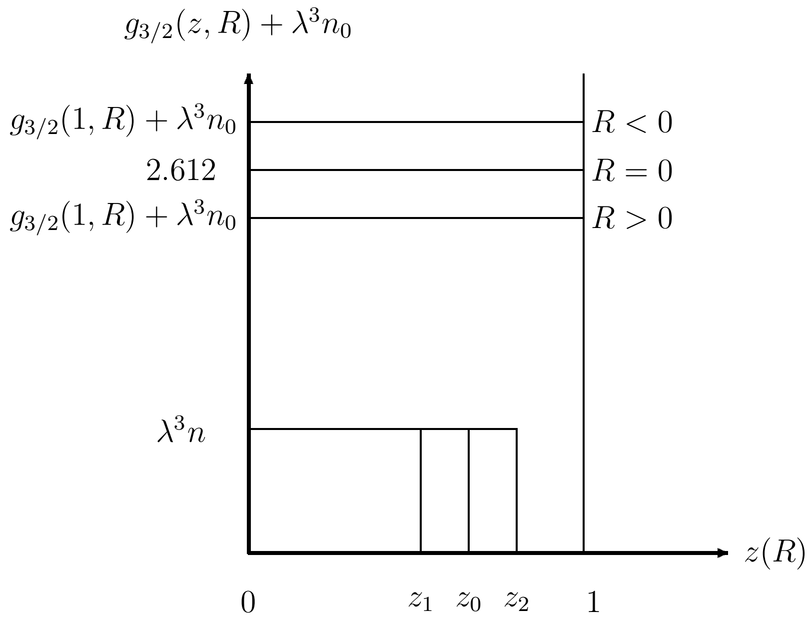


Fig.I-2 Graphical solution for bosons.

Solution of the equation (12.21) for different curvatures and fixed temperature and density z_1 for $R<0$, z_0 for $R=0$ and z_2 for $R>0$

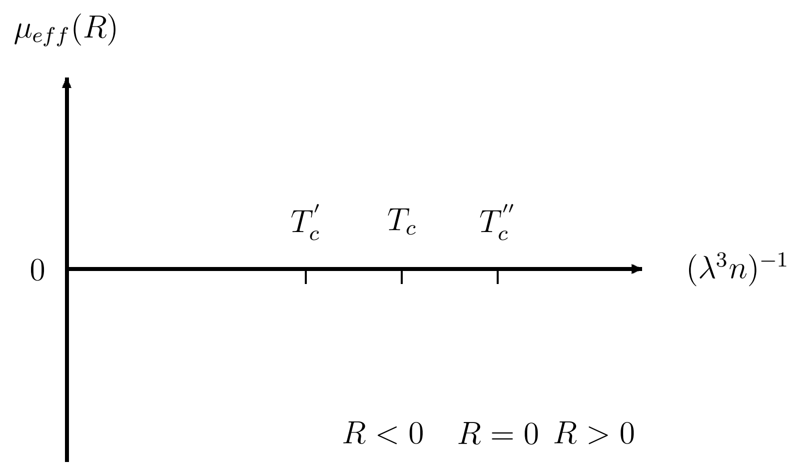


Fig.I- 3 Chemical potential $\mu_{eff}(R)$ as a functional of a curvature of space-time.

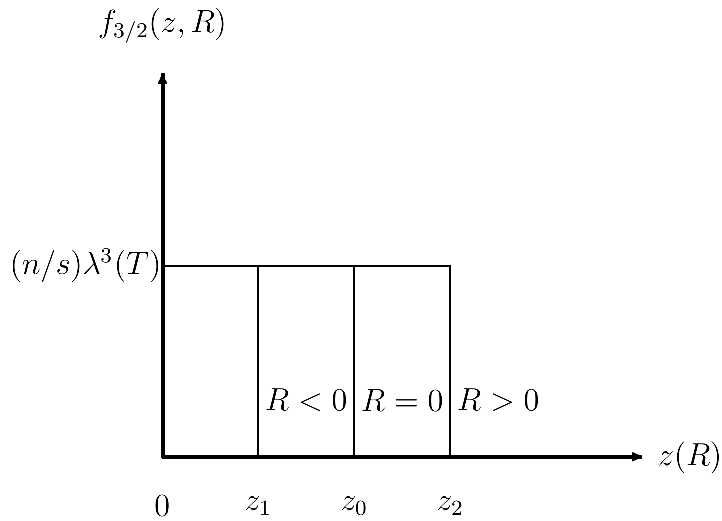


Fig.I-4 Graphical solution for fermions.

Solution of the equation (13.13) for different curvatures: z_1 for $R > 0$, z_0 for $R = 0$ and z_2 for $R < 0$

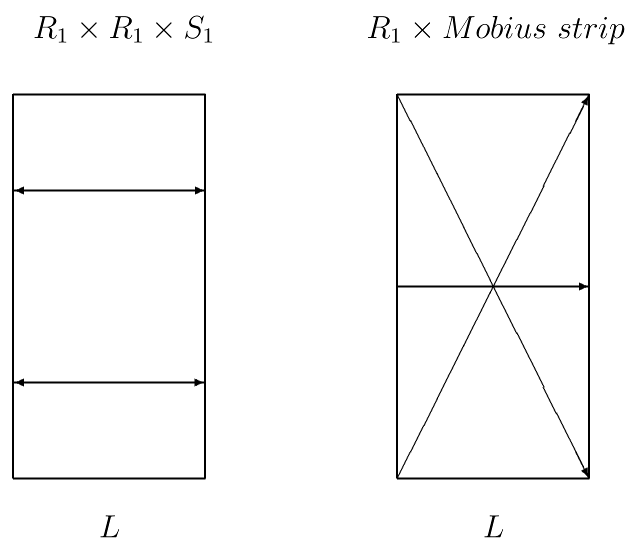


Fig. III-1 Topologies of cylinder and Mobius strip in (y, z) spaces.

Identification of the points $\psi(x, y, 0) = \psi(x, y, L)$ and $\psi(x, y, 0) = -\psi(x, y, L)$

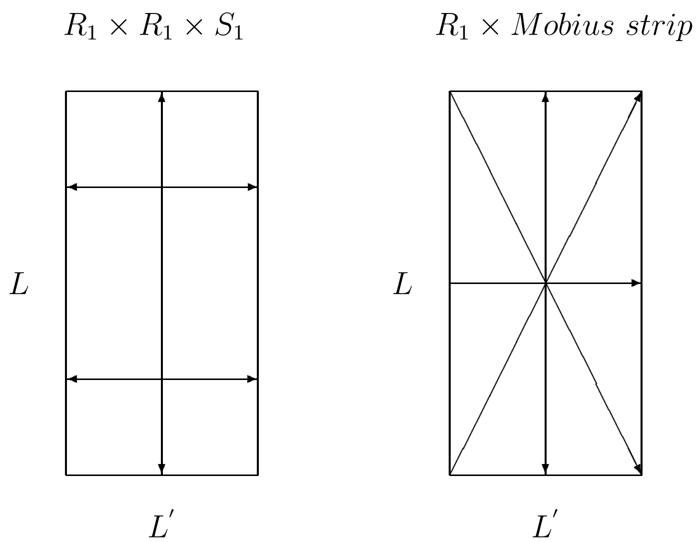


Fig. III-2 Topologies of torus and Klein bottle in (y, z) spaces.

Identification of the points $\psi(x, , 0) = \psi(x, L, L')$ and $\psi(x, 0, 0) = -\psi(x, L, L')$

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