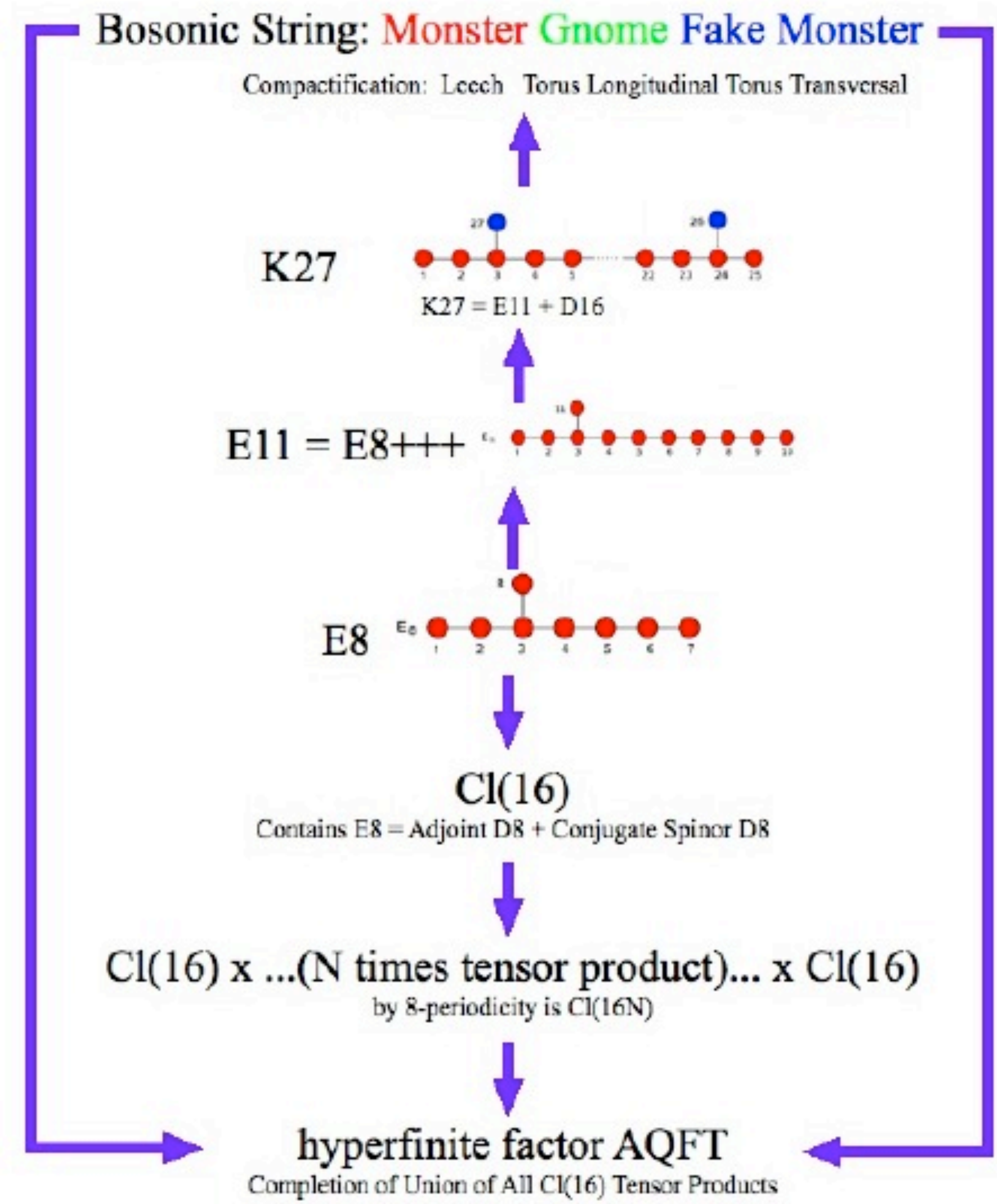
# E8 Bosonic String AQFT

Frank Dodd (Tony) Smith, Jr. - 2010



James Lepowsky in math.QA/0706.4072 said: "... the Fischer-Griess Monster M ... was constructed by Griess as a symmetry group (of order about 10^54) of a remarkable new commutative but very, very highly nonassociative, seemingly ad-hoc, algebra B of dimension 196,883. ... The Monster is the automorphism group of the smallest nontrival string theory that nature allows ... Bosonic 26-dimensional space-time ... "compactified" on 24 dimensions, using the orbifold construction V [flat] ... or more precisely, the automorphism group of the vertex operator algebra with the canonical "smallness" properties. ...".

O. Barwald, R. W. Gebert, M. Gunaydin and H. Nicolai in hep-th/9703084 said:
"... The root system of a Kac–Moody algebra is simple to describe, yet for any other but positive or positive semi-definite Cartan matrices (corresponding to finite and affine Lie algebras, resp.), the structure of the algebra itself is exceedingly complicated and not completely known even for a single example.
By contrast, Borcherds algebras can sometimes be explicitly realized as Lie algebras of physical states of some compactified bosonic string.
Famous examples are
the fake monster Lie algebra gII25,1 and the (true) monster Lie algebra g ♮ ,
arising as the Lie algebra of transversal states of a bosonic string in 26 dimensions fully compactified on a torus or a Z2-orbifold thereof, respectively ...
A distinctive feature of Lorentzian Kac–Moody algebras of "subcritical" rank (i.e., d < 26) is the occurrence of longitudinal states besides the transversal ones. This result applies in particular to the maximally extended hyperbolic algebra E10 which can be embedded into gII9,1 , the Lie algebra of physical states of a subcritical bosonic string fully compactified on the unique 10-dimensional even unimodular Lorentzian lattice II9,1. The problem of understanding E10 can thus be reduced to the problem of characterizing the "missing states" (alias "decoupled states"), i.e. those physical states in gII9,1 not belonging to E10. The problem of counting these states, in turn, is equivalent to the one of identifying all the imaginary simple roots of gII9,1 with their multiplicities. ...
gII1,1 ...[is]... the Lie algebra of physical states of a bosonic string compactified on II1,1; because of its kinship with the monster Lie algebra g ♮ which has the same root lattice, we will refer to it as the "gnome Lie algebra".
Its maximal Kac–Moody subalgebra g(A) ⊂ gII1,1 is just the finite Lie algebra A1 ≡ sl2. ... The gnome Lie algebra has not yet appeared in the literature so far, although it is possibly the simplest non-trivial example of a Borcherds algebra for which not only one has a satisfactory understanding of the imaginary simple roots, but also a completely explicit realization of the algebra itself in terms of physical string states. ...

If the fake monster Lie algebra is extremal in the sense that it contains only transversal, but no longitudinal states,
the gnome Lie algebra gII1,1 is at the extreme opposite end of the classification in that it has only longitudinal but no transversal states. ...
Hence the gnome Lie algebra represents the third example of a Borcherds algebra (besides the fake and the true monster Lie algebra), for which a complete set of simple roots is known and an explicit Lie algebra basis can be constructed. ...
The gnome Lie algebra gII1,1 , which we will investigate in this section, is the simplest example of a Borcherds algebra that can be explicitly described as the Lie algebra of physical states of a compactified string. It is based on the lattice II1,1 ...

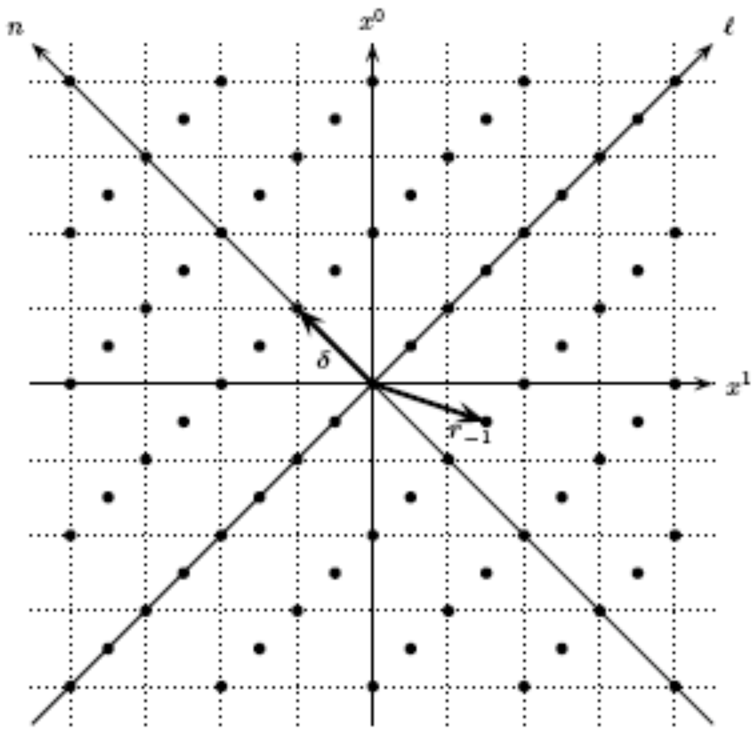



Figure 1: The Lorentzian lattice $II_{1,1}$

... as momentum lattice of a fully compactified bosonic string in two space-time dimensions. Since there aro no transversal degrees of freedom in d = 2 and only longitudinal string excitations occur, the Lie algebra of physical states may be regarded as the precise opposite of the fake monster Lie algebra in 26 dimensions which has only transversal and no longitudinal physical states. It constitutes an example of a generalized Kac–Moody algebra which is almost "purely Borcherds" in that with one exception, all its simple roots are imaginary (timelike). The gnome Lie algebra is also a cousin of the true monster Lie algebra because they both have the same root lattice, II1,1. In fact, we shall see that the gnome Lie algebra is a

Borcherds subalgebra not only of the fake monster Lie algebra but also of any Lie algebra of physical states associated with a momentum lattice that can be decomposed in such a way that it contains II1,1 as a sublattice. ...
The Weyl group of II1,1 is very simple: since we can only reflect with respect to the single root r-1, it has only two elements and is thus isomorphic to Z2 just like the Weyl group of the monster Lie algebra ...
The gnome Lie algebra is by definition the Borcherds algebra gII1,1 of physical states of a bosonic string fully compactified on the lattice II1,1. We would first like to describe its root space decomposition. ... the gnome Lie algebra looks schematically like the monster Lie algebra ...

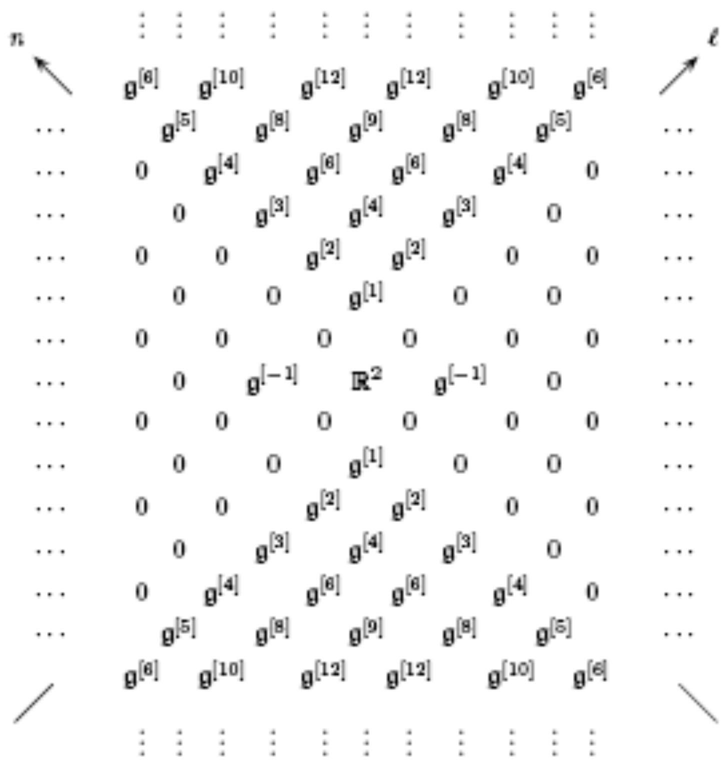



Figure 3: Root space decomposition of the gnome Lie algebra

... for increasing level the dimensions of the root spaces grow much faster than the simple multiplicities. This explains why additional imaginary simple roots are needed at every level. There is a beautiful example where this situation is rectified. The true monster Lie algebra is a Borcherds algebra which is based on the same lattice II1,1 as root lattice; but the multiplicity of a root ... is given by c(ℓn) (replacing π1(1+ℓn)) which is the coefficient of qℓn in the elliptic modular function j(q) - 744 = ∑n≥-1 c_n q^ n = q - 1 + 196884q + . . . .
for the true monster Lie algebra ... the imaginary simple roots are all of level 1.

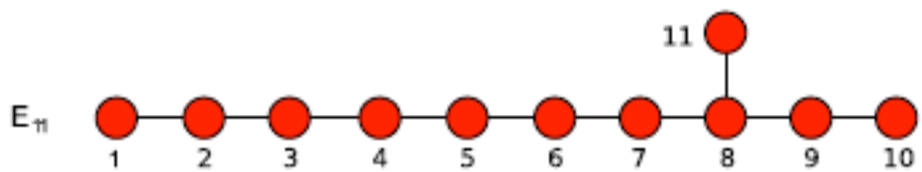
On the other hand, ... for the gnome Lie algebra ... [t]here are ... imaginary simple roots ... at level 2 or higher ... The reason for this is that the root spaces in the former example are much bigger (due to the “hidden” extra 24 dimensions of the moonshine module) ...”.

Reinhold W.Gebert and Hermann Nicolai in hep-th/9411188 said:
“... for the 26-dimensional bosonic string there is a unique choice of maximal symmetry, namely the even selfdual Lorentzian lattice II25,1 which indeed provides a “large” algebra - the infinite rank fake monster Lie algebra introduced by Borcherds ... for the 26-dimensional bosonic string, where special properties such as the no-ghost theorem play a crucial role. In this example, all missing states are under control (though not explicitly known): one has to adjoin a certain (infinite) set of photonic states as new Lie algebra generators to the ordinary Kac Moody generators in order to get a complete set of generators for the Lie algebra of physical states. The resulting algebra constitutes an example of a generalized Kac Moody algebra and has been dubbed fake monster Lie algebra ...
The imaginary simple roots corresponding to the extra generators are just the positive integer multiples of the (lightlike) Weyl vector for the lattice II25,1, and their multiplicities are equal to the number of photon states (i.e. = 24) ...
[For]... the 26-dimensional bosonic string ... the longitudinal states span the radical of the contravariant bilinear form which is divided out. Hence only transversal states survive ...”.

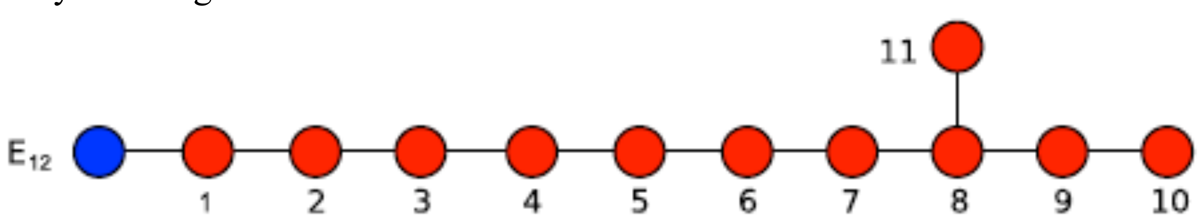
Peter West in hep-th/0208214 said:
“... for the closed bosonic string there exists a non-vanishing value of the momentum that lowers the energy and so spontaneously breaking the Lorentz and translation invariance of the theory. We interpret this as meaning that the closed bosonic string theory undergoes a spontaneous compactification. The possible relevance of the Landau theory of liquid-crystal transitions for the closed bosonic string has been discussed before. ...
although the closed bosonic string possess no supersymmetry it is thought ... to be invariant under a very powerful algebra that should determine many of its properties. As a result, the vacuum ... should break not only Lorentz and translation symmetries, but also the K27 algebra of the closed bosonic string and one may hope to find, at least in the case of breaking to the type II strings, that the relevant vacuum preserves the E11 algebra. One encouraging sign is that K27 contains the sub-algebra E11 + D16. While the first factor is clearly that required as a residual symmetry, the second factor contains a D8 sub-algebra ...”.

P. West in hep-th/0104081 said: "... The Closed Bosonic String and K27
The closed bosonic string on a torus is invariant under the fake monster Lie algebra ...
The closed bosonic string in 26 dimensions can also be formulated as a non-linear realisation ...[as]... a Kac-Moody algebra of rank 27. We call this algebra K27. ... the algebra K27 contains the algebra E11 ...".

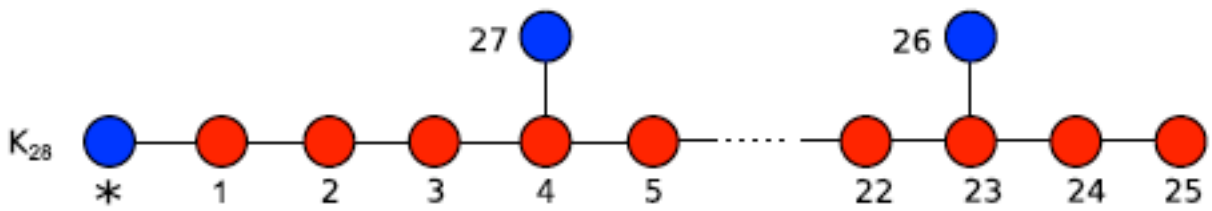
Paul P. Cook and Peter West in 0805.4451 said: "... E11 is described completely by its Dynkin diagram which is found by attaching three additional roots to the longest leg of the E8 diagram, each extra simple root having the same length as any root of E8.




The representations of E11 other than the adjoint are also interesting and of direct relevance to theoretical physics. The l1, or charge, representation of E11 is believed to contain all the brane charges of M-theory in the E11 weight lattice ... one can obtain the l1 representation of E11 by extending the E11 Dynkin diagram with a node attached by a single line to the longest leg of the E11 diagram, giving the Dynkin diagram of E12 ...




... and one then restricts to just those roots with the coefficient of the extra root, $\alpha\mathbf{\Lambda}$, set to one. In other words one decomposes E12 by the deletion of the node $\alpha\mathbf{\Lambda}$ and the l1 representation of E11 is found at level one with highest weight l1, the first fundamental weight of E11. ...
In ... the ... extension of the ... K27 = D24+++ ... algebra ...




... related to the twenty-six dimensional bosonic string ... we have twenty-six pi's related to the spacetime coordinates ...".

H. Nicolai and H. Samtleben in hep-th/0407055 said: "... it is natural to conjecture that the fermionic degrees of freedom of ...[a realistic]... theory should consequently transform as spinors (i.e. as double-valued representations) under the maximal compact subgroups of these Kac Moody groups, in accordance with the chain of embeddings of 'generalized R symmetries'
. . . ⊂ Spin(16) ⊂ K(E9) ⊂ K(E10) ⊂ . . .

... To work out the relevant spinor representations for K(E10) (and also for K(E11)) will be no easy task ...".

Francois Englert, Laurent Houart and Anne Taormina in hep-th/0203098 said:
"... The emergence of space-time fermions and of supersymmetry ... is an impressive property of the bosonic string. ...
the theory is more elegantly formulated in terms of E8 x SO(16). This formulation was in fact a crucial step, because it led to uncover not only the superstrings, but also the non-supersymmetric fermionic strings. ...
To accommodate space-time fermions in the 26-dimensional bosonic string one must meet three requirements:
a) A continuum of bosonic zero modes must be removed. This can be achieved by compactifying d = 24 − s transverse dimensions on a d-dimensional torus. This leaves s + 2 non-compact dimensions with transverse group SOtrans(s).
b) Compactification must generate an internal group SOint(s) admitting spinor representations. This can be achieved by toroidal compactification on the Lie lattice of a simply laced Lie group G of rank d containing a subgroup SOint(s). The latter is then mapped onto SOtrans(s) in such a way that the diagonal algebra sodiag(s) = diag[sotrans(s) x soint(s)] becomes identified with a new transverse algebra. In this way, the spinor representations of SOint(s) describe fermionic states because a rotation in space induces a half-angle rotation on these states.
c) The consistency of the above procedure relies on the possibility of extending the diagonal algebra sodiag(s) to the new full Lorentz algebra sodiag(s + 1, 1), a highly non trivial constraint. To break the original Lorentz group SO(25, 1) in

favour of the new one, a truncation consistent with conformal invariance must be performed on the physical spectrum of the bosonic string. Actually, states described by 12 compactified bosonic fields must be truncated, except for zero modes ...
The highest available space-time dimension accommodating fermions is therefore s + 2 = 10 ...”.

Francois Englert, Laurent Houart and Anne Taormina in hep-th/0106235 said:
“... We review the emergence of the ten-dimensional fermionic closed string theories from subspaces of the Hilbert space of the 26-dimensional bosonic closed string theory compactified on an E8 x SO(16) lattice. They arise from a consistent truncation procedure which generates space-time fermions out of bosons. ...
The derivation of these fermionic string properties from bosonic considerations alone points towards a dynamical origin of the truncation process. Space-time fermions and supersymmetries would then arise from bosonic degrees of freedom and no fermionic degrees of freedom would be needed in a fundamental theory of quantum gravity. ...
We decompose SO(16) in SO′(8) x SO(8) and truncate all states created by oscillators in the 12 dimensions defined by the E8 x SO′(8) root lattice. ...
The centre of the covering group of SO(8) is Z2 x Z2. Its four elements partition the weight lattice in four conjugacy classes (o)8, (v)8, (s)8, (c)8 isomorphic to the root lattice. The (o)8 lattice is the root lattice itself and contains the element √2α′po = (0, 0, 0, 0). The (v)8 lattice is the vector lattice whose smallest weights are eight vectors of norm one; in an orthonormal basis, these are √2α′pv = ( +/-1, 0, 0, 0) + permutations. The (s)8 and (c)8 lattices are spinor lattices whose smallest weights also have norm one and are the eightfold degenerate vectors √2α′ps,c = ( +/-1/2, +/-1/2, +/-1/2, +/-1/2 ) with even (for class (s)8) or odd (for class (c)8) number of minus signs.
The structure of the weight lattice of all SO(4m) groups is the same: in a 2m-dimensional Cartesian basis, the root lattice vectors have integer components whose sum is even (and contains the element √2α′po = 0).
The vector √2α′pv = ( +/-1, 0 ..... still has norm one
but the spinors √2α′ps,c = ( +/-1/2, +/-1/2 .....) have norm increasing with m.
The degeneracy in norm of (v)8, (s)8 and (c)8 in SO(8)
is rooted in the triality properties of the group, and the choice of a vector representation √2α′pv is a mere convention.

√2α′pv is in fact defined by

its mapping onto the representation of the SOtrans(8) group ... It is this mapping which transmutes the spinors (s)8 and (c)8 of SOint(8) to space-time spinors of the Lorentz group SO(9, 1). ...
It follows from the closure of the Lorentz algebra that states belonging to the lattices (v)8 or (o)8 are bosons while those belonging to the spinor lattices (s)8 and (c)8 are space-time fermions. These zero modes ensure the truncation consistency by selecting, in the light-cone gauge, the emission vertices of the fermionic strings as subsets of the emission vertices of the bosonic string. They may in fact be viewed as superghosts zero modes entering emission vertices in the fermionic string. We shall therefore refer to these zero modes as to ghost vectors. ...
the truncation from E8 x SO(16) to SOint(8) + ghosts transfers modular invariance from the 26-dimensional bosonic string to ten-dimensional fermionic strings ...”.

Axel Kleinschmidt, Hermann Nicolai, and Jakob Palmkvist in hep-th/0611314 said: “... we will use $\alpha$ and $\dot{\alpha}$ as SO(8) spinor and conjugate spinor indices, respectively, while the indices i, j, . . . still take the values 3, . . . , 10 as SO(8) vector indices. The chiral (8 x 8) SO(8) gamma-matrices will be denoted by $\gamma^i_{ab'}$ .
Then eight real, symmetric (16 x 16) gamma matrices of SO(9) can be written

$$\gamma^i_{IJ} = \begin{matrix} 0 & \gamma^i_{ab'} \\ \gamma^i_{a'b} & 0 \end{matrix}$$

where $\gamma^i_{a'b}$ is the transpose of $\gamma^i_{ab'}$

The first eight SO(9) gamma matrices square to one, anticommute, and define the ninth matrix ... $\gamma 2$ ...[which]... also squares to one, and anticommutes with $\gamma i$. The SO(9) gamma matrices can be extended to the ten, real, symmetric (32 x 32) gamma matrices of SO(10) ...
In these conventions, the decomposition ... of a 32 component spinor into two chiral spinors is manifest ...
Triality ... can ... be extended to SO(9) matrices ... Thus we can take as SO(16) gamma matrices ...[as]... tensor products ...[f]rom ...[which]... one can compute the non-trivial antisymmetric products ... of gamma matrices ...
the vector, spinor and conjugate spinor indices ... of SO(16) split into those of SO (8) ... according to the decompositions

$16 \rightarrow (8c, 1) \oplus (1, 8s) \rightarrow 8s \oplus 8c$, 128s → (8v, 8v)⊕(8s, 8c) → 1 ⊕ 28 ⊕ 35v ⊕ 8v ⊕ 56v,

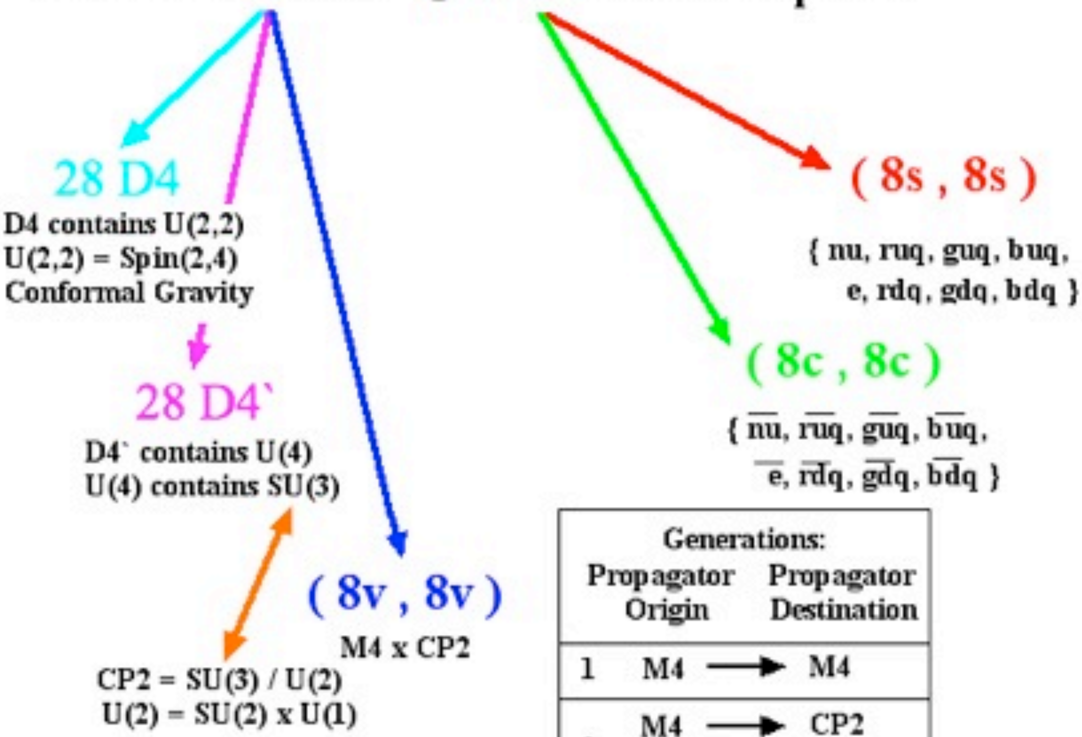
128c → (8v, 8c)⊕(8s, 8v) → 8s ⊕ 56s ⊕ 8c ⊕ 56c

of these so(16) representations under so(8)⊕ so(8), and then under the diagonal so(8) subalgebra. ...”.

In the following summary diagram,
the Higgs-Mayer diagrams are modified from
articles by Mayer and Trautman in New Developments in Mathematical Physics, 20th Universitatswochen fur Kernphysik in Schladming in February 1981 (ed. by Mitter and Pittner), Springer-Verlag 1981, which articles are:

- A Brief Introduction to the Geometry of Gauge Fields (written with Trautman);
- The Geometry of Symmetry Breaking in Gauge Theories;
- Geometric Aspects of Quantized Gauge Theories.

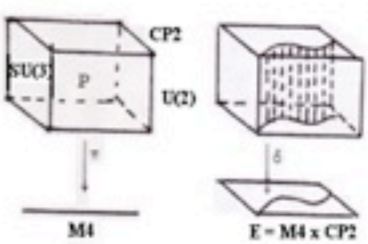
# 248 E8 = 120 D8 ⊕ 128 D8 Half Spinor




| Generations: | Propagator Origin | Propagator Destination |
|---|---|---|
| 1 | M4 | → M4 |
| 2 | M4<br>CP2 | → CP2<br>→ M4 |
| 3 | CP2 | → CP2 |

**Lagrangian:** $\int_{\text{KKspacetime}}$ gauge term + fermion term

**Higgs-Mayer:**




**Kobayashi-Nomizu:**

THEOREM 11.7. *Assume in Theorem 11.5 that* $\mathfrak{k}$ *admits a subspace* $\mathfrak{m}$ *such that* $\mathfrak{k} = \mathfrak{j} + \mathfrak{m}$ *(direct sum) and* $\mathrm{ad}(J)(\mathfrak{m}) = \mathfrak{m}$, *where* $\mathrm{ad}(J)$ *is the adjoint representation of* $J$ *in* $\mathfrak{k}$. *Then*

(1) *There is a 1:1 correspondence between the set of K-invariant connections in P and the set of linear mappings* $\Lambda_{\mathfrak{m}}: \mathfrak{m} \to \mathfrak{g}$ *such that*

$$\Lambda_{\mathfrak{m}}(\mathrm{ad}(j)(X)) = \mathrm{ad}(\lambda(j))(\Lambda_{\mathfrak{m}}(X)) \quad \text{for } X \in \mathfrak{m} \text{ and } j \in J;$$

*the correspondence is given via Theorem 11.5 by*

$$\Lambda(X) = \begin{cases} \lambda(X) & \text{if } X \in \mathfrak{j}, \\ \Lambda_{\mathfrak{m}}(X) & \text{if } X \in \mathfrak{m}. \end{cases}$$

(2) *The curvature form* $\Omega$ *of the K-invariant connection defined by* $\Lambda_{\mathfrak{m}}$ *satisfies the following condition:*

$$2\Omega_{u_0}(\tilde{X}, \tilde{Y}) = [\Lambda_{\mathfrak{m}}(X), \Lambda_{\mathfrak{m}}(Y)] - \Lambda_{\mathfrak{m}}([X, Y]_{\mathfrak{m}}) - \lambda([X, Y]_{\mathfrak{j}}) \quad \text{for } X, Y \in \mathfrak{m},$$

**The Higgs and the T-quark form a system in which the Higgs is effectively a T-quark condensate.**