Introduction to Real Clifford Algebras: from Cl(8) to E8 to Hyperfinite II1 factors

Frank Dodd (Tony) Smith, Jr. - 2013

Real Clifford Algebras roughly represent the Geometry of Real Vector Spaces of signature (p,q) with the Euclidean Space (0,q) sometimes just being written (q) so that the Clifford algebra Cl(0,q) = Cl(q).
A useful starting place for understanding how they work is to look at the most central example and then extend from it to others.
This paper is only a rough introductory description to develop intuition and is NOT detailed or rigorous - for that see the references.

Real Clifford Algebras have a tensor product periodicity property whereby

$$Cl(q+8) = Cl(q) \times Cl(8)$$

so that if you understand Cl(8) you can understand larger Clifford Algebras such as Cl(16) = Cl(8) x Cl(8) and so on for as large as you want.
So Cl(8) is taken to be the central example in this paper which has 4 parts:

How Cl(8) works - page 2

What smaller Clifford Algebras inside Cl(8) look like - page 7

How the larger Clifford Algebra Cl(16) gives E8 - page 9

How larger Clifford Algebras Cl(16N) = Cl(8(2N)) give in the large N limit a generalized Hyperfinite II1 von Neumann factor AQFT - page 14

References:

Lectures on Clifford (Geometric) Algebras and Applications
Rafal Ablamowicz, Garret Sobczyk (eds) (Birkhauser 2003)
especially lectures by Lounesto and Porteous

Clifford Algebras and Spinors
Pertti Lounesto (Cambridge 2001)

Clifford Algebras and the Classical Groups
Ian R. Porteous (Cambridge 2009)

My Introduction to E8 Physics at viXra:1108.0027

How Cl(8) works

Cl(8) is a graded algebra with
grade k corresponding to dimensionality of vectors from the origin to subspaces of 8-dim space spanned by k basis vectors of 8 orthogonal basis vectors {x1,x2,x3,x4,x5,x6,x7,x8} of the 8-dim Euclidean space.
In the following construction use only the positive basis vectors (not their mirror image negatives). That is the same as looking at only the all-positive octant of the 8 -dim Euclidean space.

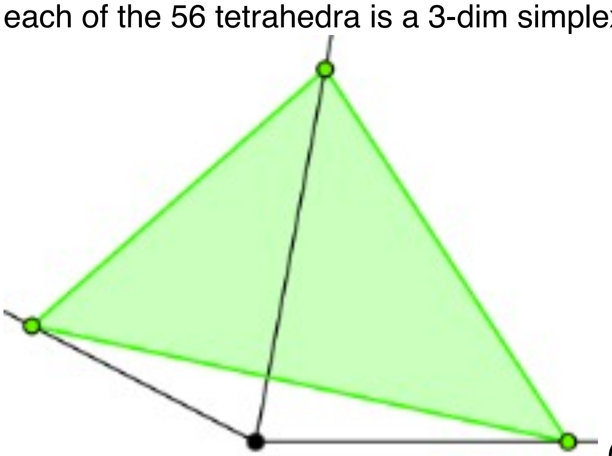
Grade:

0 - vectors from origin to itself - 0-dimensional - 1 point

1 - vectors from origin to 1 of the 8 basis vectors - 1-dim - 8 line segments
each of the 8 line segments is a 1-dim simplex
whose outer "face" is a 0-dimensional point.
These 8 basis vectors are the basis vectors of the 8-dim vector space of Cl(8).

2 - vectors from origin to pairs of the 8 basis vectors - 2-dim - 28 triangles defined by pairs of vectors
each of the 28 triangles is a 2-dim simplex (but NOT equilateral)
whose outer "face" is a 1-dimesional line segment.
These 28 bivectors (pairs of vectors) give the 28 planes of rotation of the 28-dim group Spin(8) that includes rotations of the 8-dim vector space of Cl(8).

3 - vectors from origin to triples of the 8 basis vectors - 3-dim - 56 tetrahedra defined by triples of vectors
each of the 56 tetrahedra is a 3-dim simplex (but NOT equilateral)



(image from wikipedia)
whose outer "face" is a 2-dimensional triangle (that IS equilateral).

4 - vectors from origin to 4-tuples of the 8 basis vectors - 4-dim - 70 4-simplexes defined by 4-tuples of vectors
each of the 70 4-simplexes is a 4-dim simplex (but NOT equilateral)
whose outer "face" is a 3-dimensional tetrahedron (that IS equilateral).

5 - vectors from origin to 5-tuples of the 8 basis vectors - 5-dim - 56 5-simplexes defined by 5-tuples of vectors
each of the 56 5-simplexes is a 5-dim simplex (but NOT equilateral)
whose outer "face" is a 4-dim simplex (that IS equilateral).

6 - vectors from origin to 6-tuples of the 8 basis vectors - 6-dim - 28 6-simplexes defined by 6-tuples of vectors
each of the 28 6-simplexes is a 6-dim simplex (but NOT equilateral)
whose outer "face" is a 5-dim simplex (that IS equilateral).

7 - vectors from origin to 7-tuples of the 8 basis vectors - 7-dim - 8 7-simplexes defined by 7-tuples of vectors
each of the 8 7-simplexes is a 7-dim simplex (but NOT equilateral)
whose outer "face" is a 6-dim simplex (that IS equilateral).

8 - vectors from origin to 8-tuples of the 8 basis vectors - 8-dim - 1 8-simplex defined by the unique 8-tuple of vectors
the 8-simplex is an 8-dim simplex (but NOT equilateral)
whose outer "face" is a 7-dim simplex (that IS equilateral).

The total dimension of Cl(8) is
1+8+28+56+70+56+28+8+1 = 256 = 2^8 = 16x16

The Cl(8) algebra is the algebra of 16x16 matrices of real numbers.

The product of the Cl(8) algebra is the product of 16x16 real matrices
but it also has geometric meaning.

If you multiply for example a
grade-2 element = 2-dim simplex with basis vectors { x3, x5 }
by a
grade-4 element = 4-dim simplex with basis vectors { x2, x6, x7, x8 }
then
you get a
grade-6 element = 6-dim simplex with basis vectors { x2, x3, x5, x6, x7, x8 }
BECAUSE
the Clifford Algebra product in this case acts like the exterior algebra wedge product (or the cross-product in 3-dim)
so that the product fo two independent subspaces of the Cl(8) 8-dim Euclidean space is sort of the span of both subspaces taken together

BUT

If you multiply for example a
grade-2 element = 2-dim simplex with basis vectors { x2, x5 }
by
a grade-4 element = 4-dim simplex with basis vectors { x2, x6, x7, x8 }
then
you get
a grade-4 element = 4-dim simplex with basis vectors { x5, x6, x7, x8 }
BECAUSE
the Clifford Algebra in this case acts partly like a dot-product
so that in the product of two defined-by-the-same-vector subspaces the two subspaces cancel out to zero (the common basis vector x2is cancelled).

In short, Clifford aAlgebra idescribes the geometry of vector subspaces
and
the geometry is exactly described by matrix algebras like

Cl(8) = 16x16 real matrices R(16).

The 16x16 real matrices R(16) are made up of 16 column vectors
each of which is a 16-dim vector that decomposes into two 8-dim vectors.

Since16 times an 8+8 = 16-dim column vector gives all 16x16 = 256 elements of R(16) = Cl(8)
it is useful to regard the 16-dim column vectors as fundamental square-root-type constituents of Cl(8) and to call them Cl(8) spinors.

Since the 16-dim Cl(8) spinors decompose into two 8-dim parts,
call them 8-dim +half-spinors and 8-dim -half-spinors
and denote them by **8+s** and **8-s** .

In the case of Cl(8) the grade-1 vectors are also 8-dim, denoted by **8v** , so
for Cl(8) we have a **Triality Automorphism 8+s = 8-s = 8v**
that turns out to be very useful in physics
because it gives a relation
between +half-spinors, -half-spinors, and vectors.

The equility between +half-spinors and -half-spinors gives a symmetry between fermion particles and antiparticles.

Since gauge bosons are grade-2 bivectors of which Cl(8) has 28,
the gauge boson Lagrangian dimension in 8-dim spacetime is 28.

Since in 8-dim spacetime fermions have Lagrangian dimension 7/2
the full fermion term of the Lagrangian also has dimension 28.

Therefore, in the high-energy 8-dim spacetime Lagrangian
the boson and fermion terms cancel due to the Triality Supersymmetry
of boson Lagrangian dimension = 28 = fermion Lagrangian dimension.

Once you understand the Cl(8) example
you can extend the model to Cl(N) for any N
and also extend it to spaces with any signature (p,q) for p+q = N
where p is the number of dimensions of negative signature
and q is the number of dimensions of positive signature
in the vector space over which the Clifford Algebra Cl(p,q) is defined.

Clifford Algebras for Euclidean spaces Cl(0,q) are also denoted Cl(q),
such as **Cl(8) = Cl(0,8) = R(16) and Cl(16) = Cl(0,16) = R(256)**

For some (p,q) the Clifford Algebras are matrix algebras over
complex C or quaternionic H as for example

Cl(4) = Cl(1,3) = H(2)

Cl(2) = H

Cl(1) = C

What smaller Clifford Algebras inside Cl(8) look like

Here is a table of
all Clifford Algebras Cl(p,q) smaller than Cl(8) = Cl(0,8) = R(16)

	$q \rightarrow$							
p	**R**	**C**	**H**	^{2}**H**	**H**(2)	**C**(4)	**R**(8)	^{2}**R**(8)
$\downarrow$	^{2}**R**	**R**(2)	**C**(2)	**H**(2)	^{2}**H**(2)	**H**(4)	**C**(8)	**R**(16)
	R(2)	^{2}**R**(2)	**R**(4)	**C**(4)	**H**(4)	^{2}**H**(4)	**H**(8)	**C**(16)
	C(2)	**R**(4)	^{2}**R**(4)	**R**(8)	**C**(8)	**H**(8)	^{2}**H**(8)	**H**(16)
	H(2)	**C**(4)	**R**(8)	^{2}**R**(8)	**R**(16)	**C**(16)	**H**(16)	^{2}**H**(16)
	^{2}**H**(2)	**H**(4)	**C**(8)	**R**(16)	^{2}**R**(16)	**R**(32)	**C**(32)	**H**(32)
	H(4)	^{2}**H**(4)	**H**(8)	**C**(16)	**R**(32)	^{2}**R**(32)	**R**(64)	**C**(64)
	C(8)	**H**(8)	^{2}**H**(8)	**H**(16)	**C**(32)	**R**(64)	^{2}**R**(64)	**R**(128)

from Ian Porteous's book "Clifford Algebras and the Classical Groups" (Cambridge 1995, 2009).

Some of the smaller Clifford Algebras are particularly useful in physics.

Here is how some of them fit inside Cl(8) = Cl(0,8):

Cl(8) = Cl(1,7) = R(16) = H(2) x H(2) = Cl(1,3) x Cl(1,3)
Spin(8) Triality Group and F4

Cl(2,6) = Cl(3,5) = H(8)

Cl(6) = R(8) = H x H(2)

Cl(2,4) = Cl(1,5) = H(4)
Conformal SU(2,2) Group

Cl(4) = Cl(1,3) = H(2)
Minkowsi Lorentz Group

Cl(3) = R(4) = H + H

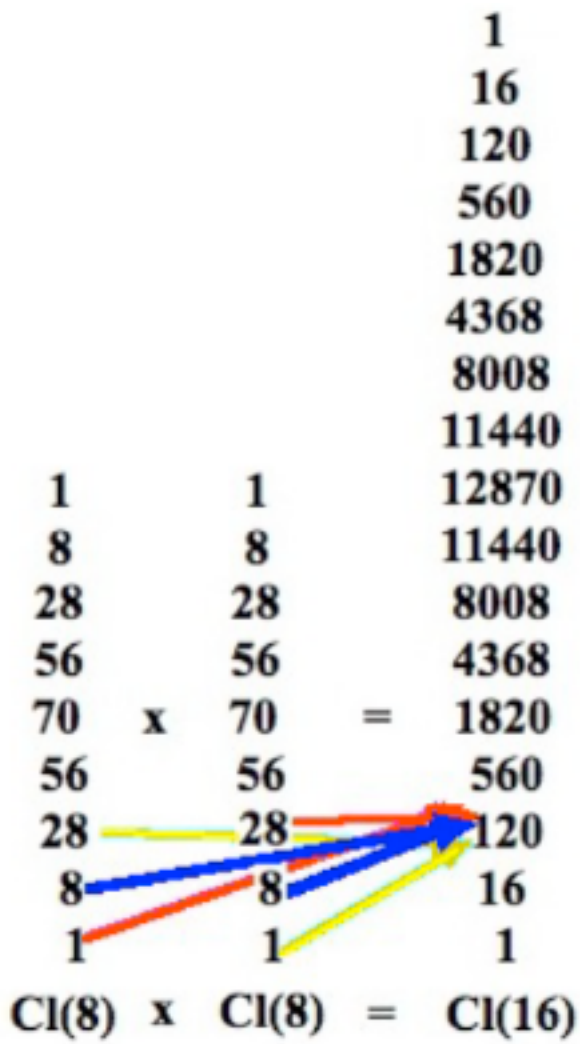
Cl(2) = H

Cl(1) = C

Cl(0) = R

How the larger Clifford Algebra Cl(16) gives E8

By Periodicity the tensor product Cl(8) x Cl(8) = Cl16 with graded structure

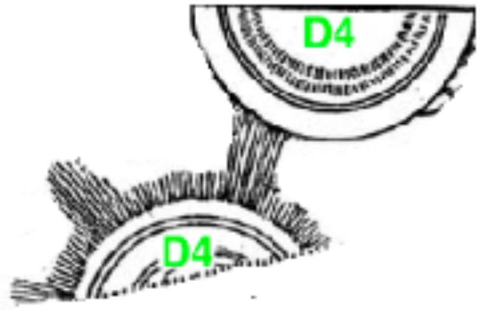


The 16-dim vector space of Cl(16) comes from Cl(8) x Cl(8)
as grade (0+1) = (1+0) = 1
and dimension 1x8 + 8x1 = 16

The 120-dim bivector space of Cl(16) comes from Cl(8) x Cl(8)
as grade (0+2) = (2+0) = (1+1) = 2
and dimension 1x28 + 28x1 + 8x8 = 28 + 28 + 64
The 28 bivectors in each of the Cl(8) generate the D4 Lie Algebra Spin(8).
The 120 bivectors in Cl(16) generate the D8 Lie Algebra Spin(16).

Therefore 120-dim D8 contains:

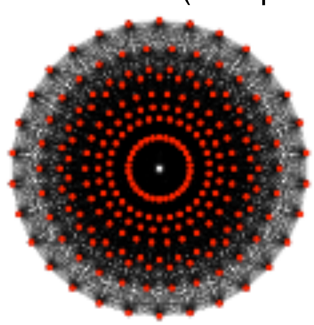
two copies of 28-dim D4



plus

a 64-dim structure that is the product of two 8-dim Cl(8) vector spaces each of which is half of the 16-dim D8 vector space so that effectively the 64-dim structure is the square of the rank 8 of D8 which is also

the rank of 248-dim E8 whose 240 Root Vectors can be seen as 8 concentric circles of 30 Root Vectors (Wikipedia image)



such that there exists the symmetric space

D8 / D4xD4 = 8x8 = 64-dim rank 8 Grassmannian

and

248-dim E8 = 120-dim D8 + 128-dim D8 +half-spinor =

= D4xD4 + 8x8 + 128-dim D8 +half-spinor D8+s

The spinors of Cl(16) = Cl(8)xCl(8) come from the spinors of Cl(8):

(**8+s** + **8-s**) x (**8+s** + **8-s**) =

= (**8+s x 8+s** + **8+s x 8-s** + **8-s x 8+s** + **8-s x 8-s**) =

= (8+s x 8+s + 8-s x 8-s) + (+ 8+s x 8-s + 8-s x 8+s) =

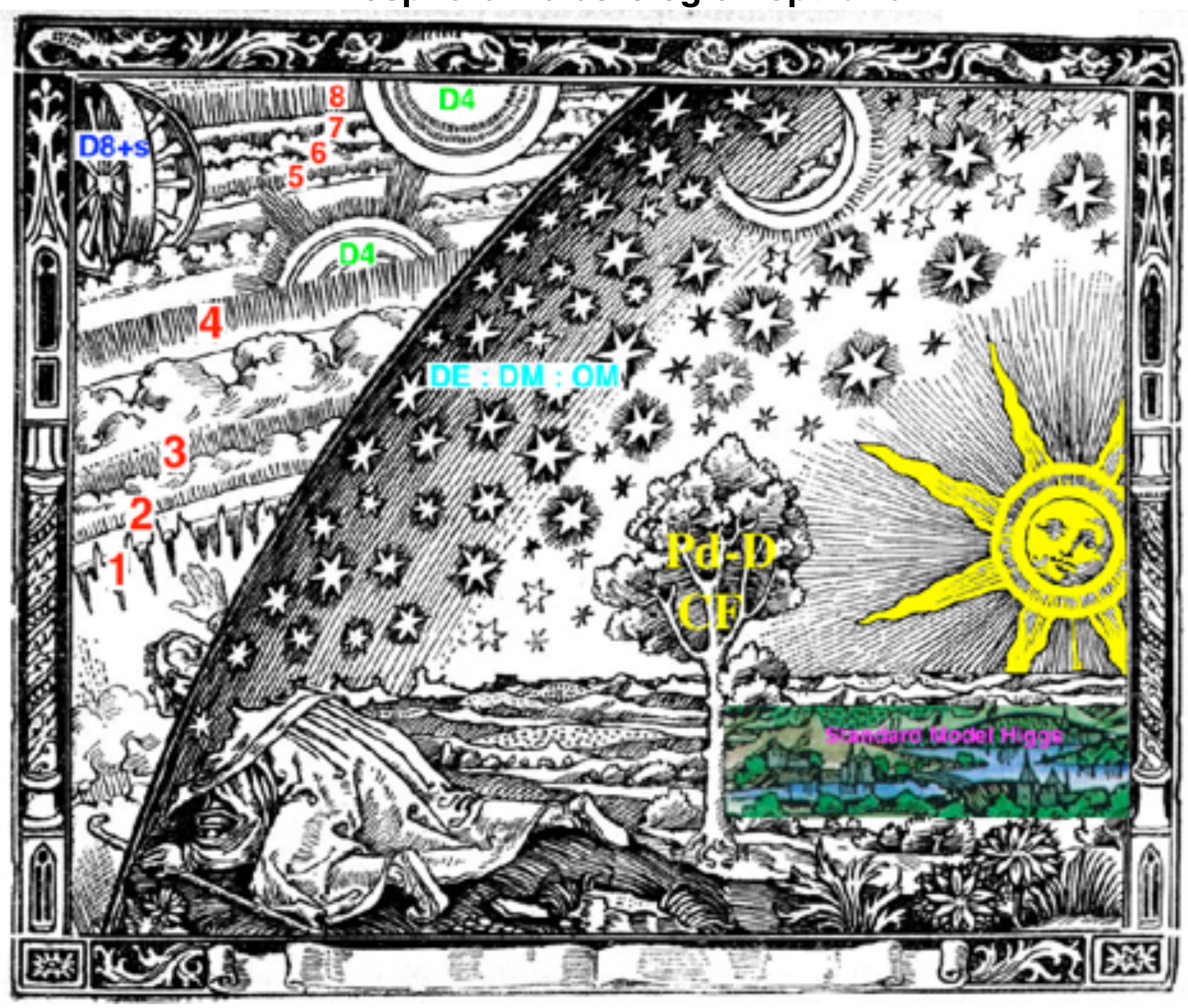
= (**64++** + **64--**) + (**64+-** + **64-+**) = 128 + 128 = 256-dim Cl(16) spinors

If you try to combine all 128+128 = 256 of the Cl(16) D8 spinors
with the 120-dim Cl(16) D8 Lie Algebra you will see that they will fail
to make a nice Lie Algebra
but
if you take only the (**64++** + **64--**) = 128-dim Cl(16) D8 +half-spinors D8+s

and combine them with the 120-dim Cl(16) D8 Lie Algebra
they DO form the 128+120 = 248-dim E8 Lie Algebra.
Although E8, like all Lie Algebras, can be written in terms of commutators,
the 128-dim D8 +half-spinor part of E8 can be written as anticommutators,
a property that E8 in Cl(16) inherits from F4 in Cl(8). Therefore,
the 120-dim D8 part of E8 physically represents boson and vector
spacetime with commutators
and
the 128-dim D8 +half-spinor part of E8 physically represents fermions with
anticommutators. Further, since it is made up of (**64++** + **64--**)
it represents 8 spacetime components of 8 fermion particles
plus 8 spacetime components of 8 fermion antiparticles.
The 8 first-generation fermion types are
electron , red up quark , green up quark , blue up quark ;
blue down quark , green down quark , red down quark, neutrino

Second and Third fermion generations, Higgs, and 3 mass states of Higgs and Tquark emerge as consequences of the Octonion / Quaternion transition from 8-dim Spacetime of the Inflationary Era of our Universe to 4-dim Physical Spacetime + 4-dim CP2 Internal Symmetry Space.

Could the D4xD4 + 8x8 + 128-dim D8 +half-spinor D8+s structure of E8 have been depicted by Flammarion on page 163 of his 1888 book "L'Atmosphere Meteorologie Populaire" ?



Flammarion's Naive Missionary Explorer sees
the intersection of Terrestrial Physics and AstroPhysics as a window to
the Realm of Terrestrial-AstroPhysics Unification through E8 Physics.

As to Terrestrial Physics: a Standard Model Higgs has been observed by the LHC near Lake Geneva which looks like this part of the Flammarion Engraving (colorization from image on goodnewsfromdrjoe blog)

and

progress has been made toward understanding Palladium-Deuterium Cold Fusion, planting a seed that can grow into a productive Tree of Energy

As to AstroPhysics: Hot Fusion was known to be the Energy Source of the Sun

and

our Universe was shown to have a ratio DE : DM : OM

of Dark Energy to Dark Matter to Ordinary Matter roughly 3/4 : 1/5 : 1/20

How larger Clifford Algebras Cl(8N) give in the large N limit a generalized Hyperfinite II1 von Neumann factor AQFT

As to Clifford Algebras larger than Cl(0,8)
there is periodicity theorem that
Cl(0,n+8) = M(16,Cl(0,n)) of all 16x16 matrices
whose entries are from Cl(0,n)
and Cl(p,q+4) = Cl(p,q) x Cl(0,4) = Cl(p,q) x H(2)
and Cl(p,q) = Cl(p+4,q-4)
and Cl(p+8,q) = Cl(p,q) x Cl(0,8) = Cl(p,q) x H(2) x H(2) = Cl(p,q) x R(16)
and Cl(p,q+8) = Cl(p,q) x Cl(0,8) = Cl(p,q) x H(2) x H(2) = Cl(p,q) x R(16)

whereby
the tensor product of n copies of Cl(8)

Cl(8) x…(n times tensor product)…x Cl(8) = Cl(8n)

and
any really large Clifford Algebra can
be embedded in the tensor product of a lot of Cl(8) Clifford Algebras.

Since the E8 Physics classical Lagrangian is Local,
it is necessary to patch together Local Lagrangian Regions
to form a Global Structure describing
a Global E8 Algebraic Quantum Field Theory (AQFT).
Mathematically,
this is done by embedding E8 into the Cl(16) Clifford Algebra and
using a copy of Cl(16) to represent each Local Lagrangian Region.

A Global Structure is then formed
by taking the tensor products of the copies of Cl(16).
Due to Real Clifford Algebra 8-periodicity, Cl(16) = Cl(8)xCl(8)
and any Real Clifford Algebra, no matter how large, can be embedded in a
tensor product of factors of Cl(8), and therefore of Cl(8)xCl(8) = Cl(16).

Just as the completion of the union of all tensor products
of 2x2 complex Clifford algebra matrices produces
the usual Hyperfinite II1 von Neumann factor
that describes **creation and annihilation operators**
on the fermionic Fock space over C^(2n) (see John Baez's Week 175),
we can take
the completion of the union of
all tensor products of Cl(16) = Cl(8)xCl(8)
to produce a generalized Hyperfinite II1 von Neumann factor that gives
a natural Algebraic Quantum Field Theory structure for E8 Physics.

In each tensor product Cl(16) x ... x Cl(16)
each of the Cl(16) factors represents a distinct Local Lagrangian Region.
Since each Region is distinguishable from any other,
each factor of the tensor product is distinguishable
so that the AQFT has Maxwell-Boltzmann Statistics.
Within each Local Lagrangian Region Cl(16) there lives a copy of E8.
Each 248-dim E8 has indistinguishable boson and fermion particles.
The 120-dim bosonic part has commutators and Bose Statistics
and the 128-dim fermionic part has anticommutators and Fermi Statistics.

The E8 Local Classical Lagrangian structure has a direct correspondence
with the AQFT Creation-Annihilation Quantum Operator structure
by the correspondence between
E8 and its **Contraction semidirect product A7 + h_92** where
the Heisenberg algebra h_92 = 28 + 64 + 1 + 64 + 28 is made up of
the central 1
plus
28 + 28 for **creation and annihilation of 28 D4 gauge bosons**
with 16 of the 28 giving U(2,2) for Conformal Gravity
and 12 of the 28 giving the gauge bosons of the Standard Model
plus
64 + 64 for **creation and annihilation of** 8x8 = 64 components
of **8 fundamental fermions** with respect to 8-dim spacetime
and
the **central A7 + 1 of the semidirect product is U(8)** within D8 of E8
that describes **8 position x 8 momentum** dimensions of 8-dim spacetime
with (4+4)-dim Kaluza-Klein Quaternionic structure.

AQFT Possibility Space for E8 Physics must include, for each vertex of each E8 lattice, the 2^240 possible ways that each of the 240 vertices of its First Shell Root Vectors can be either 0 or 1 (off or on, inactive or active, etc).
Since 2^240 = Cl(240) = Cl(8x30) = Cl(8) x...(30 times tensor product)...x Cl(8)
the First Shell Possibility Space is the tensor product of 30 copies of Cl(8) or,
equivalently since Cl(8)xCl(8) = Cl(16) which contains E8 as bivectors + half-spinors,
the First Shell Possibility Space is the tensor product of 15 copies of Cl(16)

Since the Second Shell has 2160 vertices, its Possibility Space must include
Cl(2160) = Cl(8x270) = Cl(8) x...(270 times tensor product)...x Cl(8)
so the Second Shell Possibility Space has 135 copies of Cl(16).

Here (from Conway and Sloane, Sphere Packings, Lattices, and Groups, 3rd ed, Springer 1999) are

Table 4.10. The first 8 shells of E_8.

Norm	Number	Vectors
0	1	0^8.
2	240	$1^2 0^6$, $E\,(1/2)^8$.
4	2160	20^7, $1^4 0^4$, $D\,(3/2)(1/2)^7$.
6	6720	$21^2 0^5$, $1^6 0^2$, $E\,(3/2)^2(1/2)^6$.
8	17520	$2^2 0^6$, $21^4 0^3$, 1^8, $D\,(3/2)^3(1/2)^5$, $E\,(5/2)(1/2)^7$.
10	30240	310^6, $2^2 1^2 0^4$, $21^6 0$, $D\,(5/2)(3/2)(1/2)^6$, $E\,(3/2)^4(1/2)^4$.
12	60480	$31^3 0^4$, $2^3 0^5$, $2^2 1^4 0^2$, $E\,(5/2)(3/2)^2(1/2)^5$, $D\,(3/2)^5(1/2)^3$.
14	82560	3210^5, $31^5 0^2$, $2^3 1^2 0^3$, $2^2 1^6$, $D\,(7/2)(1/2)^7$, $E\,(5/2)^2(1/2)^6$, $D\,(5/2)(3/2)^3(1/2)^4$, $E\,(3/2)^6(1/2)^2$.
16	140400	40^7, $321^3 0^3$, 31^7, $2^4 0^4$, $2^3 1^4 0$, $E\,(7/2)(3/2)(1/2)^6$, $D\,(5/2)^2(3/2)(1/2)^5$, $E\,(5/2)(3/2)^4(1/2)^3$, $D\,(3/2)^7(1/2)$.

so Possibility Space for each E8 lattice contains tensor products of MANY Cl(16) copies
15 + 135 + 420 + 1,095 + 1,890 + 3,780 + 5,160 + 8,775 + ... more from beyond 8 shells

If you take the Union of all those Tensor Products of copies of Cl(16)
(each of which contains E8 = bivectors + half-spinors)
and then take the Completion of that,
you will get a generalization of the Hyperfinite II1 von Neumann factor algebra
but
even that is only 1 part (corresponding to one E8 Lattice) of the realistic AQFT of E8 Physics.

To get the full realistic Algebraic Quantum Field Theory of E8 Physics
you need
the Superposition of 8 Cl(16)-E8 generalized Hyperfinite II1 von Neumann factors:
7 for the 7 independent E8 Integral Domain Lattices
+ 1 Kirmse E8 Lattice (not Integral Domain)